FINANCIAL RISK MANAGEMENT: TOPICS ON BANK’S SUPERVISION AND RISK MODELING

By

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To my father, Markos
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ΠΕΡΙΛΗΨΗ

Οι κεντρικές τράπεζες παγκοσμίως έχουν ως κύρια αρμοδιότητα την εποπτεία του χρηματοπιστωτικού συστήματος, συμπεριλαμβανομένων και των τραπεζών. Για την αποδοτικότερη επίτευξη του σκοπού της εποπτείας, οι κεντρικές τράπεζες ελέγχουν την εφαρμογή της νομοθεσίας, η οποία αφορά στο χρηματοπιστωτικό σύστημα και επιπλέον είναι επιφορτισμένες με το καθήκον να σχεδιάζουν, να υλοποιούν και να παρακολουθούν την ύπαρξη του δυνατού πιστότητα εφαρμογή του κανονιστικού πλαισίου με απότερο σκοπό την εύρυθμη λειτουργία και τη σταθερότητα του συστήματος. Εκτός από τις υπόλοιπες μεθόδους που έχουν στην εργαλειοθήκη τους οι ρυθμιστικές αρχές, υπάρχει και η χρήση των εσωτερικών υποδειγμάτων, ως βάση για την καθοδήγηση των κεφαλαιακών απαιτήσεων κάθε τράπεζας, οι οποίες απορρέουν από τις επιχειρηματικές δραστηριότητες των τραπεζών. Η ανάπτυξη αυτών των εσωτερικών μοντέλων υπόκειται σε αυστηρά ποσοτικά και ποιοτικά πρότυπα, όπως αυτά ορίζονται από τις κεντρικές τράπεζες. Ο λόγος για τον οποίο οι εποπτικές αρχές προσπαθούν να επιβάλλουν την χρήση μοντέλων μέτρησης κινδύνων είναι η αποφυγή μιας μελλοντικής μαζικής διαρροής καταθέσεων κάθε τράπεζας (“bank run”), γεγονός το οποίο θα απέβαινε καταστροφικό και θα απειλούσε τη σταθερότητα συνολικά του χρηματοπιστωτικού συστήματος. Προκειμένου να μετριασθεί αυτός ο κίνδυνος είναι επιβεβλημένο από κάθε τράπεζα να διατηρεί διαφορετικά και σταθερά επίπεδα αποθεματικών για διαφορετικές μορφές τραπεζικών καταθέσεων.

Η Επιτροπή της Βασιλείας για την Τραπεζική Εποπτεία (στην Ευρώπη) και το Federal Deposit Insurance Corporation (στις ΗΠΑ) έχουν ως βασικό σκοπό την προώθηση της ασφαλείας και της ευρωστίας των χρηματιστηρίων ιδρυμάτων, συμπεριλαμβανομένων και των τραπεζών. Η επίτευξη αυτού του στόχου γίνεται με την εφαρμογή των κανονιστικών μεθόδων για την εντοπισμό, την παρακολούθηση και την αντιμετώπιση των πάσης φύσεως κινδύνων, οι οποίοι απειλούν τα χρηματοπιστωτικά ιδρύματα.

Στο αντικείμενο της τραπεζικής εποπτείας τα τελευταία χρόνια με αφορμή τη χρηματοπιστωτική κρίση του 2007 και της συνεχιζόμενης οικονομικής κρίσης, όπως
αυτή εξελίχθηκε μέχρι και σήμερα, υπάρχει μεγάλο ενδιαφέρον για τα τρία ακόλουθα θέματα:(α) Ποιοι είναι οι παράγοντες κινδύνου που περιλαμβάνουν χρήσιμες πληροφορίες για την πρόβλεψη της χρεοκοπίας μιας τράπεζας;(β) Ποις μπορεί κανείς να αναπτύξει ένα σύστημα έγκαιρης προειδοποίησης που θα μπορούσε να δώσει έγκαιρα σημάδια μιας επερχόμενης κρίσης;(γ) Ποις μπορεί να μοντελοποιηθεί η στοχαστική συμπεριφορά των δυνητικών απωλειών, οι οποίες απορρέουν από τις επενδύσεις υψηλού κινδύνου μιας τράπεζας και επηρεάζουν άμεσα την ασφαλή λειτουργία του ιδρύματος;

Η παρούσα διδακτορική διατριβή εξετάζει και φιλοδοξεί να απαντήσει στα τρία παραπάνω ερωτήματα, παρέχοντας διάφορα εργαλεία που θα μπορούσαν να αξιοποιηθούν από τις εποπτικές αρχές και τους χρηματοπιστωτικούς οργανισμούς για την εύρυθμη λειτουργία του τραπεζικού συστήματος. Επιπλέον, συμβάλλει στην περιοχή της μοντελοποίησης των χρηματοοικονομικών κινδύνων με την εισαγωγή νέων οικογενειών κατανομών, οι οποίες μπορούν να αξιοποιηθούν επωφελώς στην περιγραφή των χαρτοφυλακίων ή στις αποδόσεις των επενδύσεων των χρηματοπιστωτικών ιδρυμάτων.

Ο πρώτος κύριος άξονας της διατριβής είναι η σύγκριση των δύο τελευταίων δεκαετιών όσον αφορά τη διαχρονική εξέλιξη του δείκτη κεφαλαιακής επάρκειας των αμερικάνικων τραπεζών. Τα δεδομένα που χρησιμοποιήθηκαν για την ανάλυση προήλθαν από όλες τις εμπορικές τράπεζες των ΗΠΑ και καλύπτουν την χρονική περίοδο 1992-2009. Χρησιμοποιώντας μια Μαρκοβιανή αλυσίδα, οι τράπεζες χωρίστηκαν σε 5 κατηγορίες, κάθε μια από τις οποίες εκφράζει την αντίστοιχη κατηγοριοποίηση του FDIC, όπως αυτό έχει διακριτικοποιηθεί το δείκτη κεφαλαιακής επάρκειας. Κατόπιν, υπολογίστηκαν οι αντίστοιχες πιθανότητες μετάβασης μεταξύ όλων των δυνατών καταστάσεων, στις οποίες μπορεί να βρεθεί μια τράπεζα, συμπεριλαμβανομένης της κατάστασης χρεοκοπίας. Το πλεονέκτημα αυτής της προσέγγισης είναι ότι επιτρέπει την ποσοτικοποίηση των πιθανοτήτων μετάβασης των τραπεζών για όλες τις πιθανές καταστάσεις (δηλαδή, για όλο το «φάσμα κεφαλαιοποίησης»), μεταξύ των οποίων είναι και η κατάσταση χρεοκοπίας, η οποία εποπτικά παρουσιάζει ιδιαίτερο ενδιαφέρον. Η ανάλυσή μας επιτρέπει την εκτίμηση των δεικτών κινητικότητας και των μέτρων απόστασης, τα οποία μπορούν να χρησιμοποιηθούν για τον προσδιορισμό της βελτίωσης ή χειροτέρευσης της
κεφαλαιοποίησης του τραπεζικού συστήματος συνολικά. Αυτή η πληροφορία είναι ιδιαίτερα χρήσιμη τόσο για τα ίδια τα πιστωτικά ιδρύματα, όσο (ίσως περισσότερο) για την εποπτική αρχή. Αυτή η προσέγγιση επιτρέπει τον αποτελεσματικό εντοπισμό των ομοιοτήτων και των διαφορών όσο αφορά την κεφαλαιοποίηση των τραπεζών για όλη την περίοδο κατά την οποία ισχύει το κανονιστικό πλαίσιο της Βασιλείας. Από την ανάλυση των δεδομένων προκύπτουν αρκετά σημαντικές διαφορές στην ανατομία των δύο περιόδων (1992-2000 έναντι 2001-2009). Η προτεινόμενη προσέγγιση εισάγει μια μεθοδολογία που θα μπορούσε να εφαρμοστεί στο μέλλον, τόσο για τη σύγκριση δεδομένων χρονικών περιόδων στις οποίες υπάρχει η υποψία ότι είναι διαφορετικές, καθώς και για την ανάπτυξη ενός συστήματος έγκαιρης προειδοποίησης με σκοπό τον έγκαιρο εντοπισμό επερχόμενων συστημικών κρίσεων στον τραπεζικό τομέα.

Ο δεύτερος στόχος της διατριβής έγκειται στην ανακάλυψη εκείνων των παραγόντων CAMEL, οι οποίοι φέρουν σημαντική πληροφορία για την εκ των προτέρων ύπαρξη των συνθηκών, οι οποίες θα μπορούσαν να οδηγήσουν στο απότελεσμα μέλλον σε πτώχευση. Χρησιμοποιούνται δεδομένα που καλύπτουν την περίοδο 2000-2011 και αφορούν όλες τις εμπορικές τράπεζες των ΗΠΑ, οι οποίες είναι ασφαλισμένες από το FDIC, θα διερευνηθεί πόσα τρίμηνα πριν από την χρεοκοπία, οι παράγοντες αυτοί περιέχουν χρήσιμα μηνύματα για την ύπαρξη δομικών αστοχιών, οι οποίες εγείρουν υποψίες για την εύρυθμη λειτουργία των τραπεζών. Η καινοτομία της ανάλυσης είναι ότι εστιάζεται στους παράγοντες εκείνους που αφορούν στις ομάδες των τραπεζών που έχουν έκθεση στο ίδιο επίπεδο του υπό εξέταση παράγοντα κινδύνου κάθε φορά. Οι στατιστικές μεθοδολογίες που χρησιμοποιούνται περιλαμβάνουν την Πολυμεταβλητή Ανάλυση (Ανάλυση κυρίων συνιστωσών, Διαχωριστική Ανάλυση και Correspondence Analysis) και τη λογιστική παλινδρόμηση. Από τα ευρήματα της ανάλυσης γίνεται αντιληπτό ότι προκύπτει μια αλληλουχία καταστάσεων που υπήρχε στο ίδιο επίπεδο του υπό εξέταση παράγοντα κινδύνου κάθε φορά. Οι στατιστικές μεθοδολογίες που χρησιμοποιούνται περιλαμβάνουν την Πολυμεταβλητή Ανάλυση (Ανάλυση κυρίων συνιστωσών, Διαχωριστική Ανάλυση και Correspondence Analysis) και τη λογιστική παλινδρόμηση. Από τα ευρήματα της ανάλυσης γίνεται αντιληπτό ότι προκύπτει μια αλληλουχία καταστάσεων που προηγούνται της χρεοκοπίας μιας τράπεζας με βάση τους παραγόντες CAMEL. Πιο συγκεκριμένα, μπορεί να δημιουργηθεί ένας κύκλος ζωής της τράπεζας (πριν από τη χρεοκοπία της), ο οποίος χωρίζεται σε 3 στάδια: Το πρώτο στάδιο είναι 4-5 χρόνια πριν από τη χρεοκοπία, το δεύτερο 3 χρόνια πριν από τη χρεοκοπία και τέλος, το τρίτο και τελευταίο στάδιο είναι 1 έτος πριν από χρεοκοπία. Στο στάδιο Ι, τρεις από τους παράγοντες κινδύνου CAMEL περιέχουν σημαντική ισχύ στην πρόβλεψη των μελλοντικών χρεοκοπιών. Στο στάδιο ΙΙ, το
σύνολο των χρήσιμων προγνωστικών παραγόντων διπλασιάζεται, ενώ κατά το τρίτο στάδιο οι περισσότεροι από τους παράγοντες κινδύνου (10 από 11) στέλνουν ισχυρά σήματα ώστε επίκειται η χρεοκοπία της τράπεζας. Τα αποτελέσματα αυτά είναι ιδιαίτερα χρήσιμα τόσο για τις ιδιες τις τράπεζες, αλλά κυρίως για την εποπτική αρχή, όταν προκύψει η ανάγκη να προσδιοριστεί ποσοτικά η κατάσταση συγκεκριμένων ομάδων τραπεζών -επίφοβοι για τη σταθερότητα του χρηματοπιστωτικού συστήματος- για διαφορετικούς χρονικούς ορίζοντες.

Συνεχίζοντας, η διατριβή ασχολείται με το πρόβλημα της δημιουργίας νέων μοντέλων κατανομών που μπορούν να χρησιμοποιηθούν επωφελώς για την περιγραφή αρκετών χρηματοοικονομικών στοιχείων (π.χ. αποδόσεις των μετοχών, συναλλαγματικές ισοτιμίες, κλπ.). Εισάγονται στη βιβλιογραφία δύο γενικεύσεις της Λογιστικής Κατανομής, οι οποίες έχουν εξαιρετικά καλή προσαρμογή σε πραγματικά χρηματοοικονομικά δεδομένα. Το σημείο-κλειδί για τη δημιουργία του νέου μοντέλου είναι η γνωστή ιδιότητα της Λογιστικής κατανομής. Πιο συγκεκριμένα είναι γνωστό ότι ο μετασχηματισμός logit της αθροιστικής συνάρτησης κατανομής της είναι ένας γραμμικός συνδυασμός αυτής. Ενα θεματικό χαρακτηριστικό της νέας οικογένειας κατανομών, είναι ότι μπορεί να σχηματίσει μια ποικιλία μοντέλων με μια πληθώρα μορφών (μονοκόρυφες, δικόρυφες, συμμετρικές και μη). Σε αυτή τη διατριβή, εκτός από την εισαγωγή των νέων κατανομών που προαναφέραμε, μελετώνται τα χαρακτηριστικά τους και προτείνονται αρκετές μέθοδοι για την εκτίμηση των παραμέτρων τους. Επιπλέον γίνεται ανάπτυξη συγκεκριμένων μεθόδων για την εκτίμηση των γνωστών μέτρων κινδύνου που χρησιμοποιούνται στα χρηματοοικονομικά, δηλαδή της Αξίας σε Κίνδυνο (Value at Risk) και του Expected Shortfall. Τέλος, αποτυπώνεται ο τρόπος με τον οποίο οι νέες κατανομές μπορούν να χρησιμοποιηθούν για να παραχθούν τέτοια παραμετρικά μοντέλα, τα οποία περιγράφουν ικανοποιητικά χρηματοοικονομικά δεδομένα που παρουσιάζουν μη κανονική (ακραία & ασταθής) συμπεριφορά. Πιο συγκεκριμένα, τέτοια χρηματοοικονομικά στοιχεία είναι η συναλλαγματικές ισοτιμίες κυρίως νομισμάτων έναντι του ευρώ και οι αντίστοιχες αποδόσεις των χρηματιστηριακών δεικτών της χώρας στην οποία χρησιμοποιούνται.
Bank supervisory agencies are responsible for monitoring the financial conditions of commercial banks, enforcing related legislation and regulatory policy. Besides the tools enforced by the regulators, banks are usually enforced to develop their own internal models as a basis for measuring their market risk capital requirements, subject to strict quantitative and qualitative standards. The risks of a bank run may become disastrous and cause instability to the whole financial system. In order to mitigate a risk it is required by each bank to maintain differing levels of reserves for various forms of bank deposits.

The Basel Committee on Banking Supervision (in Europe) and the Federal Deposit Insurance Corporation (in USA) promote the safety and soundness of depository institutions, including banks by appropriately identifying, monitoring, and addressing risks.

Three issues of major importance in bank supervision and monitoring the financial conditions of the banking sector are the following: (a) which risk factors contain useful information for bank failure? (b) how can one develop an early warning system that could give timely signs of an on-coming crisis? (c) how can one model the stochastic behavior of loss related to risky investments that directly affect the safe operation of the bank?

The present thesis contributes to the above three issues. It provides several tools that could be exploited by regulators and financial organizations for bank supervision; moreover, it contributes to the risk modelling area by introducing new families of loss distributions that can be profitably exploited in describing financial assets or returns.

A first task of the thesis is the comparison of the two past decades in terms of US banks' capitalization mobility and persistence. The empirical analysis is performed on a panel dataset whose cross section consists of all US commercial banks and its time series dimension of the period 1992-2009. Using a Markov Chain setup whose states correspond to the FDIC buckets (based on the FDIC discretized version of Capital Ratio), we estimate the transition probabilities between capitalization buckets;
including the default state. The benefit of this approach is that it allows the quantification of migration probabilities of banks across the capitalization spectrum, among which those related to the default state are of special interest. Our analysis permits estimating direction-specific mobility indices and distance measures that can used for identifying improvements or deteriorations of capitalization in the bank sector as a system. This information is particularly useful both for market participants, but especially for the regulator. This approach allows tracing similarities and/or differences effectively over the whole period in which the Basel requirements are in place. Our empirical analysis documents several substantial differences in the anatomy of two periods (90's vs 00's). The suggested approach introduces a methodology that could be practiced in the future both for comparing specific economic periods as well as for highlighting possible developing trends, thereof getting an early warning of on-coming systemic crisis in the bank sector.

Another contribution of the thesis is in identifying which CAMEL factors exhibit ex ante long term forecasting power for bank failures. Using a panel dataset covering the period 2000:q1-2011:q4 whose cross section consists of all US commercial banks insured by the FDIC, we explore how many quarters prior to failure, they may contain useful signals for the excess vulnerability of banks. The novelty of the empirical analysis is that it focuses on signals pertaining not to specific banks, but rather to groups of banks exposed to certain level of the risk factors. The statistical tools practiced in this task include, Correspondence Analysis, regression logit models, Discriminant Analysis and Principal Component Analysis. Our empirical findings indicate that, according to the group of CAMEL factors that send out signals for bank failure, we may break up the bank life cycle (prior to failure) to 3 stages: 4-5 years prior to failure (Stage I), 3 years prior to failure (Stage II) and finally 1 year prior to failure (Stage III). In Stage I, three of the CAMEL risk factors contain significant forecasting power for future failures. In Stage II the set of significant predictors is doubled while in Stage III most of the risk factors (10 out of 11) send out strong signals for bank failure. These results are particularly useful both for market participants, but especially for the regulator, when the need arises to quantify the excess vulnerability of specific groups of banks across different time horizons.
On another aspect, the thesis addresses the problem of establishing new distribution models that can be profitably used for describing several financial assets (e.g. stock returns, exchange rates etc.). We introduce two generalizations of the Logistic distribution which offer quite remarkable adaptability in real data arising in finance. The key point for the creation of the new model is the well-known property of the Logistic distribution that the logit transformation of its cumulative distribution function is a linear function. An important feature of the new family of distributions, is that it accommodates a variety of models with a plethora of shapes (unimodal, bimodal, symmetric and non–symmetric). In this thesis, besides the introduction of the new distributions, we study its characteristics, we discuss several methods for estimating its parameters and offer some formulae for evaluating the Value at Risk and Expected Shortfall metrics. In addition, we illustrate how the new distributions can be used to gain accurate parametric models for several financial data, namely for Euro foreign exchange reference rates, and stock returns of major international capital markets.
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PART I. LITERATURE REVIEW
CHAPTER 1. REGULATION AND BANK SUPERVISION-
LITERATURE REVIEW

INTRODUCTION

Bank supervisory agencies are responsible for monitoring the financial conditions of commercial banks, enforcing related legislation and regulatory policy. Besides the tools utilized by the regulators, banks are usually enforced to develop their own internal models as a basis for measuring their market risk capital requirements.

The Basel Committee on Banking Supervision (in Europe) and the Federal Deposit Insurance Corporation (in USA) promote the safety and soundness of depository institutions, including banks by appropriately identifying, monitoring, and addressing risks.

The present thesis addresses several issues pertaining to the safe operation of banks as well as to the regulatory process. More specifically it introduces and studies several tools that could be exploited by regulators and bank supervising organizations under the Basel and the US supervisory scheme. These tools could also be practiced by the banks themselves in order to accomplish the requirements set by the supervisory authorities.

In this Chapter, we give an overview of the regulatory framework under which European and US banks must operate so that the financial stability of the system be ensured. We also give details concerning the operation and the guidelines set by the Basel Committee on Banking Supervision as well as how the Federal Deposit Insurance Corporation achieves its mandate.

Next we perform a literature review of the methods employed for forecasting bank defaults, focusing on several risk factors related to bank characteristics. We introduce the necessary tools for quantifying risks and present in brief the two most popular risk measures used in Finance, Value at Risk and Expected Shortfall. Finally we substantiate the usefulness of probability models in describing loss distributions and asset returns.
1.1 BASEL COMMITTEE ON BANKING SUPERVISION (BCBS)

1.1.1 Structure of BCBS

The Basel Committee on Banking Supervision (BCBS hereafter) is the primary global standard-setter for the prudential regulation of banks and provides a forum for cooperation on banking supervisory matters. It does not possess any formal supranational authority, and therefore its decisions do not have legal force. On the contrary, the BCBS relies on its members’ commitments to achieve its mandate. The latter is to strengthen the regulation, supervision and practices of banks worldwide with the purpose of enhancing financial stability.

In order for the BCBS to achieve its mandate, it is involved in a series of activities. Thus, it exchanges information on developments in the banking sector and financial markets, to help identify current or emerging risks for the global financial system. As it wants to promote common understanding and to improve cross-border cooperation, it shares supervisory issues, approaches and techniques among its members. Among its endeavors are the establishment and the promotion of global standards for the regulation and supervision of banks as well as of guidelines and sound practices. Moreover, the BCBS addresses regulatory and supervisory gaps that pose risks to financial stability. It also monitors the implementation of its standards in member countries and beyond. The purpose of the aforementioned actions is to ensure their timely, consistent and effective implementation, as well as to contribute to a “level playing field” among internationally-active banks. In order to promote the implementation of BCBS standards, guidelines and sound practices beyond BCBS member countries; it consults with central banks and bank supervisory authorities, which are not its members and coordinates; it also cooperates with other financial sector standard setters, as well as international bodies, particularly those involved in promoting financial stability.

There are two kinds of organizations in the scheme of the BCBS: a) members and b) observers. Members are organizations with direct banking supervisory authority and central banks, whereas observers are other organizations called upon by the Chairman. Both membership and observer status are been reviewed periodically by the BCBS (BCBS, Charter,(Jan 2013)).
Members have a lot of responsibilities such as to work together in order to achieve the mandate of the BCBS and to promote financial stability. Moreover, they have to continuously enhance their quality of banking regulation and supervision, as well as, to contribute to the development of BCBS standards, guidelines and sound practices. Members should implement and apply BCBS standards in their regional jurisdictions within the predefined timeframe established by the Committee. They also have to undergo BCBS’s reviews to assess the consistency and the effectiveness of domestic rules and supervisory practices in relation to BCBS standards. Last but not least, while they participate in BCBS work and decision-making, they have to contribute to the promotion of global financial stability and not solely to their national interests.

The BCBS is not independent, actually it reports to the Group of Governors and Heads of Supervision (GHOS). The latter, endorses all the major decisions of the BCBS. Among other things, the GHOS appoints the BCBS Chairman from among its members, approves the BCBS Charter and any amendments that are added to it and provides general direction for the BCBS work programme (www.bis.org/bcbs).

The internal organizational structure of the BCBS consists of the Chairman, the Secretariat, the Committee, working groups and task forces. The role and the responsibilities of the three aforementioned entities are as follow:

The Chairman directs the work of the Committee in accordance with the BCBS mandate. As it is mentioned before GHOS appoints the Chairman. In case that the BCBS Chairman ceases to be a GHOS member before the end of its term, the GHOS will appoint a new Chairman. Until then, the Chairman’s functions are assumed by the Secretary General. In fact, each time the Chairman is unable to fulfill one of its responsibilities the Secretary General can act on its behalf. Some of the responsibilities of the Chairman are to convene and chair Committee meetings as well as to monitor the progress of the BCBS work programme and to provide operational guidance between meetings. Also, it has to carry forward the decisions and directions of the Committee and when appropriate, it reports to the GHOS. Of course, it is the principal spokesperson for the BCBS and represents it externally (BCBS, Charter, (Jan 2013)).

The Secretariat is provided by the Bank for International Settlements (BIS) and is staffed mainly by professional personnel. This staff is mainly on temporary
secondment from BCBS members. The Secretariat supports the work of the Committee, the Chairman and the groups. Not only does it facilitate coordination across groups, working groups and task forces but also acts as a liaison between BCBS members and non-member authorities. One of its main responsibilities is to ensure timely and effective information flow to all BCBS members. It maintains the BCBS records, administers the BCBS website and deals with the correspondence of the BCBS, whereas it supports the cooperation between the BCBS and other institutions. Of course, a lot of other functions can be assigned to the Secretariat by the Committee and the Chairman (BCBS, Charter, (Jan 2013)).

The Committee consists of representatives of all BCBS members and observers. These representatives have to be senior officials of their organizations. It is actually the ultimate decision-making body of the BCBS with responsibility for ensuring that the latter’s mandate is achieved. Among its responsibilities are to establish and promote the BCBS standards, guidelines and sound practices. Moreover, it has to establish and disband groups, working groups and task forces; approving and modifying their mandates; and monitoring their progress. The Committee can also develop, guide and monitor the BCBS work programme within the general direction provided by GHOS and also can recommend to the GHOS amendments to the BCBS Charter. Finally, the Committee can decide on the organizational regulations governing its activities (Bank for International Settlements, BCBS Charter, January 2013).

1.1.2. A Brief History of the Basel Committee

In 1974 the central bank governors of the G10 countries (a group of industrial countries consulting and cooperating on economic, monetary and financial matters) established a Committee on Banking Regulations and Supervisory Practices, which was later renamed as the Basel Committee on Banking Supervision (BCBS). The reason for this establishment was the breakdown of the Bretton Woods system of managed exchange rates in 1973 as well as the bankruptcy of the Bankhaus Herstatt, West Germany and the Franklin National Bank of New York. From the very beginning, the aim of the BCBS was to enhance financial stability by improving supervisory knowhow and the quality of banking supervision worldwide.
Nowadays, the BCSB includes 27 jurisdictions and reports to an oversight body, the Group of Central Bank Governors and Heads of Supervision (GHOS), which comprises central bank governors and (non-central bank) heads of supervision from member countries. Its meetings have been held regularly three or four times a year (www.bis.org/bcbs).

In 1975 the BCBS issued the “Concordat”, a paper in which it set out the rules by which host and parent (or home) supervisory authorities should share supervisory responsibility for banks’ foreign branches, subsidiaries and joint ventures. This was in accordance with the main aim of the BCBS’s work of closing the gaps in international supervisory coverage. The ultimate goal was that no foreign banking establishment would escape supervision; and supervision would be equal and consistent across member jurisdictions (Concordat, (1975)).

The Concordat was revised and re-issued quite a few times until 1992, when it was reformulated as minimum standards. These standards were communicated to all banking supervisory authorities, not only to the old ones but also to the new who joined the BCBS after its invitation.

Finally in 2012 the BCBS’s “Core principles for effective banking supervision” was published. In its final form, after the endorsement of supervisors from 140 countries, this document embraces 29 principles, covering supervisory powers, the need for early intervention and timely supervisory actions; supervisory expectations of banks, and compliance with supervisory standards.

1.1.3. The Road to Regulation

A. BASEL I: The first Basel Accord.

In July 1988, the Basel Capital Accord (or the 1988 Accord), a capital measurement system, was approved and released to banks by the G10 Governors. Actually, this was an important step towards an international minimum capital standard, which until then mainly depended on each individual bank. Due to the Accord, every country with international banking business had to succeed for a minimum capital ratio of capital to risk-weighted assets of 8% by the end of 1992 (Basel I: (July 1988)).
The Capital Accord was expanded by the BCBS several times. Thus, in 1991 an amendment was issued in order to give greater precision to the definition of general provisions, whereas in 1995 an addition was inserted regarding the recognition of the effects of bilateral netting of banks’ credit exposures in derivative products and the expansion of the matrix of add-on factors. The last addition was inserted after the publication by the G-30 (an influential international body consisting of senior representatives of the private, public sectors and academia) of a seminal report addressing for the first time so-called off-balance-sheet products, like derivatives, in a systematic way and the understanding of the banking industry that it clearly had the need for a proper risk management of these new products. In April 1996, a new amendment of how BCBS members intended to recognize the effects of multilateral netting was issued. Finally, the BCBS issued the ‘Market Risk Amendment to the Capital Accord’, in January 1996. The most important issue of this document is that banks are allowed to use internal value-at-risk models as a basis for measuring their market risk capital requirements, subject to strict quantitative and qualitative standards.

B. BASEL II: The new capital framework

In June 2004, the BCBS released the Revised Capital Framework in order to replace the 1988 Accord. The main issue of this new Basel Accord is credit risk, meaning that banks can use a finer, more risk-sensitive approach to assessing the risk of their credit portfolios. In other words, banks are allowed to use internal and/or external credit-rating systems wherever they deem appropriate a more advanced, so-called internal-ratings-based approach. Moreover, a significant issue of Basel II is the consideration of operational risk as a new risk class (Basel II: (June 2006)). The Revised Capital Framework, known as “Basel II”, is structured in three pillars, namely:

• Pillar 1: Minimum Capital Requirements. Actually this develops and expands the standardized rules set out in the 1988 Accord by adopting more risk sensitive minimum capital requirements for banking organizations;

• Pillar 2: Supervisory Review Process. Bank must assess solvency versus risk profile and actually supervisory review of an institution’s capital adequacy and internal assessment process; and

• Pillar 3: Market Discipline. This regards the effective use of disclosure as a lever to strengthen market discipline and encourage sound banking practices.
The main options of Pillar 1 can be divided to three different risks. *The Credit Risk*-unstructured exposures: which include three approaches namely standardized, internal rating-based foundation and internal rating based-advanced. *The Credit Risk-securitization:* including also three approaches like standardized, rating based and internal assessment as well as a supervision formula. And finally, *the Operational risk*, with the basic indicator approach, the standardized and the advanced measurement approaches. It is worth to mention that the Complexity of all these approaches increases as the Capital consumption decreases. In other word, the various approaches are designed to produce lower capital requirements when moving from simple to more elaborated options (in fact, this is not always the case, depending on the particular risk profile of the bank). This generates an incentive for banks to increase their risk management standards.

Pillar 2 is probably the most important of the three pillars, even if less text is devoted to it in Basel II. One could laconically summarize pillar 2 by the following commandment from the regulators (Balthazar L., (2006)):

‘Evaluate all your risks, cover them with capital, and we will check what you have done.’

In fact, three different kind of risks have to be handled under Pillar 2: i) risks that are not fully captured by Pillar 1 (i.e. concentration risk in credit risk), ii) risks that are not covered by pillar 1 (i.e. interest rate risk in the banking book, strategic risk, reputation risk, etc) and iii) risks external to the bank (business cycle effects) (Basel II: (June 2006)).

Pillar 3 intends to establish *market discipline* through appropriate both of qualitative and quantitative disclosures (Table 1.1) of risk measures, as well as other information relevant to risk management, like the scope of application, capital, risk exposures, and risk assessment processes. In particular, banks will have to offer greater insight into the adequacy of their capitalization. In fact, Pillar 3 relies on market participants to actively monitor the banks in which they have an interest (Basel II: (June 2006)).

In June 2006, the Committee in cooperation with the International Organization of Securities Commissions (IOSCO), the international body of securities regulators, published a document regarding the treatment of banks’ trading books under the new framework.
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<tr>
<td>Securitization</td>
<td>• Bank’s objectives in relation to securitization activity&lt;br&gt;• Regulatory capital approaches&lt;br&gt;• Bank’s accounting policies for securitization activities</td>
<td>• Total outstanding exposures securitized by the bank&lt;br&gt;• Losses recognized by the bank during current period&lt;br&gt;• Aggregate amount of securitization exposures retained or purchased</td>
</tr>
<tr>
<td>Market risk (internal models)</td>
<td>• General qualitative disclosure: strategies and processes, scope and nature of risk measurement system</td>
<td>• High, mean, and low Value at Risk values over the reporting period and period end&lt;br&gt;• Comparison of Value at Risk estimates with actual gains/losses</td>
</tr>
<tr>
<td>Operational risk</td>
<td>• Approach(es) for operational risk capital assessment for which the bank qualifies&lt;br&gt;• Description of the Advanced Measurement Approach, if used by the bank</td>
<td>• Book and fair value of investments&lt;br&gt;• Publicly traded/private investments&lt;br&gt;• Cumulative realized gains (losses) arising from sales and liquidations</td>
</tr>
<tr>
<td>Equities</td>
<td>• Policies covering the valuation and accounting of equity holdings in banking book&lt;br&gt;• Differentiation between strategic and other holdings</td>
<td>• Increase (decline) in earnings or economic value for upward and downward rate shocks broken down by currency (as relevant)</td>
</tr>
</tbody>
</table>
| Interest rate risk in banking book (IRRBB) | • Assumptions regarding loan pre-payments and behavior of non-maturity deposits, and frequency of IRRBB measurement | * Received from Balthazar L. (2006), From Basel 1 to Basel 3: The Integration of State-of-the-Art Risk Modeling in Banking Regulation, Palgrave MacMillan, N.Y.*
C. Towards BASEL III

In September 2008, the same month that Lehman Brothers collapsed, the Committee issued *Principles for sound liquidity risk management and supervision*. This came as a demand of fixing the results of the combination of three factors which manifests in the mispricing of credit and liquidity risk, and excess credit growth. These factors were

i) the banking sector which entered the crisis with too much leverage and inadequate liquidity buffers

ii) the poor governance and risk management and

iii) inappropriate incentive structures.

In November 2010, the new capital and liquidity standards, were endorsed at the G20 Leaders Summit in Seoul and in mid December 2010 were set out by the BCBS to form the ‘*Basel III: International Framework for liquidity risk measurement, standards and monitoring*’. In this framework the three pillars established by Basel II are not only revised and strengthened but also are extended with innovations as follows (Bank for international settlements, June 2011):

- additional layer of common equity – the capital conservation buffer – that, when breached, restricts payouts of earnings
- restricts payouts of earnings to help protect the minimum common equity requirement;
- countercyclical capital buffer, which places restrictions on participation by banks in system-wide credit booms with the aim of reducing their losses in credit busts;
- proposals to require additional capital and liquidity to be held by banks whose failure would threaten the entire banking system (i.e. too big to fail);
- leverage ratio – a minimum amount of loss-absorbing capital relative to all of a bank’s assets and off-balance-sheet exposures regardless of risk weighting;
- liquidity requirements – a minimum liquidity ratio, intended to provide enough cash to cover funding needs over a 30-day period of stress; and a longer-term ratio intended to address maturity mismatches over the entire balance sheet; and
- additional proposals for systemically important banks, including requirements for augmented contingent capital and strengthened arrangements for cross-border supervision and resolution.

From this point of view, it is obvious that Basel III is primarily related to the risks of a bank run. Thus, to mitigate such risk it is required by every bank to maintain differing levels of reserves for various forms of bank deposits, whereas Basel I and Basel II are explicitly related to the required level of loss reserves that must be held.
by banks for various classes of loans, other financial instruments, as well as other assets that they might have. This means, that Basel III rules - for the most part - work alongside to the guidelines known as Basel I, II and simultaneously supersede them. Basel III introduced two more capital buffers, namely mandatory capital conservation buffer of 2.5% and a discretionary counter-cyclical buffer. These two new capital buffers would allow national regulators to require up to an additional 2.5% of capital, during periods of high credit growth. Basel III also introduced a minimum leverage ratio and two liquidity ratios. The leverage ratio was calculated by dividing Tier 1 capital (the core measure of a Bank’s financial strength from a regulator’s point of view) by the bank's average total consolidated assets. On the other hand, the Liquidity Coverage Ratio is supposed to require a bank to hold sufficient high-quality liquid assets to cover its total net cash outflows over 30 days, whereas the Net Stable Funding Ratio is to require the available amount to exceed the required amount of stable funding over a one-year period of extended stress (Balthazar L, (2006)).

The new strengthened definition of capital introduced by Basel III will be fully implemented by the end of 2017, whereas its requirements were introduced in 2013. Capital instruments that no longer qualify as non-common equity Tier 1 capital or Tier 2 capital (supplementary capital, includes a number of important and legitimate constituents of a bank's capital base) will be phased out over 10 years beginning January, 1st, 2013 (Basel Committee on Banking Supervision (BCBS) Bank for International Settlements (BIS): [www.bis.org/bcbs](http://www.bis.org/bcbs)).

The schedule for the establishment of the minimum capital requirements, the higher minimums for common equity and Tier 1 capital, which will become effective at the beginning of 2015 (starting from 2013) is as:

- At the beginning of 2013, the minimum common equity and Tier 1 requirements increased from 2% and 4% levels to 3.5% and 4.5%, respectively.
- At the beginning of 2014, the minimum common equity and Tier 1 requirements will be 4% and 5.5%, respectively.
- The final requirements for common equity and Tier 1 capital will be 4.5% and 6%, respectively, beginning in 2015.

The 2.5% capital conservation buffer, which will comprise common equity and is in addition to the 4.5% minimum requirement, will be phased in progressively starting on 1 January 2016, and will become fully effective by January, 1st, 2019.
The leverage ratio will also be phased in gradually. The test (the so-called “parallel run period”) began in 2013 and will run until 2017, with a view to migrating to a Pillar 1 treatment on January 1st, 2018 based on review and appropriate calibration.

According to an Organisation for Economic Co-operation and Development (OECD) study, which was released on February 17th, 2011, the medium-term impact of Basel III implementation on GDP growth would be in the range of −0.05% to −0.15% per year. Economic output would be mainly affected by an increase in bank lending spreads, as banks pass a rise in bank funding costs, due to higher capital requirements, to their customers. Capital requirements that will come in effect in 2019 (7% for the common equity ratio, 8.5% for the Tier 1 capital ratio) could increase bank lending spreads by about 50 basis points. The estimated effects on GDP growth assume no active response from monetary policy. To the extent that monetary policy would no longer be constrained by the zero lower bound, the Basel III impact on economic output could be offset by a reduction (or delayed increase) in monetary policy rates by about 30 to 80 basis points (OECD: www.oecd.org). All the aforementioned important dates of the shift from Basel I to Basel III are indicated in the Table 1.2.

**Table 1.2: From Basel I to Basel III**

<table>
<thead>
<tr>
<th></th>
<th>Issued</th>
<th>Implemented</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basel I</td>
<td>July 1988</td>
<td>Dec 1992</td>
</tr>
<tr>
<td>Market Risk Amendment</td>
<td>Dec 1996</td>
<td>Dec 1997</td>
</tr>
<tr>
<td>Basel II</td>
<td>June 2004</td>
<td>Dec 2006</td>
</tr>
<tr>
<td>Basel II Advanced Approaches</td>
<td></td>
<td>Dec 2007</td>
</tr>
<tr>
<td>Revised Securitisation and Trading Book Rules</td>
<td>July 2009</td>
<td>Dec 2011</td>
</tr>
<tr>
<td>Basel III</td>
<td>Nov 2010*</td>
<td>Jan 2013-Jan 2019</td>
</tr>
</tbody>
</table>

* G 20 endorsement of Basel III

### 1.2 FEDERAL DEPOSIT INSURANCE CORPORATION (FDIC)

#### 1.2.1 A Brief History of FDIC and Its Structure

The Federal Deposit Insurance Corporation (FDIC) was created in 1933 in the U.S., as an independent agency of the federal government, in response to more than 12,000 bank failures that occurred from 1920 through 1930. The establishment of the FDIC came to restore stability and confidence in the banking system. Nowadays, it
can be argued, that FDIC preserves and promotes public confidence in the U.S. financial system by insuring deposits in banks and thrift institutions; by identifying, monitoring and addressing risks to the deposit insurance funds; and by limiting the effect on the economy and the financial system when a bank or thrift institution fails (FDIC, http://www.fdic.gov/ (5/5/2014)).

In more detail, the FDIC directly examines and supervises more than 4,500 banks and savings banks for operational safety and soundness, more than half of the institutions in the banking system. In the U.S. there are two kinds of banks, those that are chartered by the states and they have the choice of whether to join the Federal Reserve System and those that are chartered by the federal government. The FDIC acts as a federal regulator of banks that are chartered by the states and do not join the Federal Reserve System. Moreover, the FDIC is the back-up supervisor for the remaining insured banks and thrift institutions (FDIC(1998)).

In addition, the FDIC examines the conformity of the banks with consumer protection laws, like the Fair Credit Billing Act, but also with the Community Reinvestment Act which requires banks to abide strictly by the credit needs of the communities they conduct its operations.

When a bank or thrift institution fails, the FDIC acts immediately in order to protect insured depositors using several tools – while the preferable one is to sell deposits and loans of the failed institution to another institution. In this way, the FDIC transfers seamlessly the customers of the failed institution to the assuming institution. Banking institutions are closed by their chartering authority, namely the state regulator, or the Office of the Comptroller of the Currency.

The FDIC’s capital originates from premiums that banks and thrift institutions pay to FDIC for deposit insurance coverage and also its investments in U.S. Treasury securities and receives no Congressional appropriations. The FDIC insures only deposits, like the traditional types of bank accounts - checking, savings, and certificates of deposit (CDs). It does not insure securities, mutual funds, annuities, life insurance policies, stocks and bonds or similar types of investments that banks and thrift institutions may offer. Today, FDIC insures deposits as high as $9 trillion in virtually every bank and thrift in U.S. (Kahn C.M., Santos, J.A.C, (2001)).

There are several types of financial contracts, which provide insurance against default and are closely related to deposit insurance or guarantees of non-deposit bank
debt. Actually, the deposit insurance is exactly like Credit Default Swap (CDS) contract written on a deposit. The main difference with a CDS contract is that the bank pays the CDS premium to the protection seller on the behalf of the depositor. Deposit insurance and CDS guarantees on bank debt have risk characteristics that differ in important ways relative to most other forms of insurance, such as life and property/casualty insurance. The risks of deposit insurance losses due to bank failures cannot be diversified by pooling the risks of many banks together because deposit insurance loss claims are not independent or uncorrelated events. Bank failures are linked to macro-economic conditions, which tend to create financial distress to more than one bank at the same time. This is not surprising, since bank assets consist largely of real estate, commercial, and consumer loans, which experience higher default rates during economic downturns. Thus, bank failures and deposit insurance losses seem to rise during recessions and decline otherwise, so that they bear ‘systematic’ risk.

Nowadays, the Board of Directors of the FDIC is comprised of a five-member committee, among which are the Comptroller of the Currency and the director Consumer Financial Protection Bureau. All of them are appointed by the President of USA and confirmed by the US Senate, with no more than three being from the same political party. The FDIC headquarters are in Washington, D.C., but conducts much of its business by having six regional offices, one temporary satellite office and by field offices around the country and employs more than 7,000 people. Lately, the FDIC Board of Directors approved the establishment of three different advisory committees in order to facilitate its actions. These are: (a) the Advisory Committee on Economic Inclusion (ComE-IN), which was chartered in November 2006. The role of this Committee is to provide the FDIC with advice and recommendations on important initiatives focused on expanding access to banking services by underserved populations. This may include reviewing basic retail financial services, such as check cashing, money orders, remittances, stored value cards, short-term loans, savings accounts, and other services that promote asset accumulation by individuals and financial stability: (b) the FDIC Advisory Committee on Community Banking, which was established on May of 2009 and its aim is to provide the FDIC with advice and guidance on a broad range of important policy issues, impacting small community banks throughout the country, as well as the local communities they serve, with a
focus on rural areas; and finally (c) the FDIC Systemic Resolution Advisory Committee, established on July 21st, 2010 with main purpose to provide advice and recommendations on a broad range of issues regarding the resolution of Systemically Important Financial Institutions (SIFI’s) pursuant to the Dodd-Frank Wall Street Reform and Consumer Protection Act.

As a deposit insurer, the FDIC has an interest in structuring its system to best achieve the goals of ensuring depositor confidence and financial stability. Having that in mind, it should be stressed that each banking crisis confirms the important role that deposit insurance plays in this regard. In the U.S., each banking crisis also has come with the mandate for the FDIC to operate in ways that achieves these goals while, at the same time, mitigating the hazards associated with deposit insurance and protecting the interests of taxpayers. These competing interests can be observed in all aspects of a deposit insurer's operations, including funding arrangements.

1.2.2. The Deposit Insurance Fund

As it is mentioned before, the FDIC promotes the safety and soundness of insured depository institutions and the U.S. financial system by identifying, monitoring, and addressing risks, which have an impact to the deposit insurance funds (DIF). In fact, the Financial Institutions Reform, Recovery, and Enforcement Act of 1989 established two separate deposit insurance funds, the Bank Insurance Fund (BIF) and the Savings Association Insurance Fund (SAIF) and placed them under the FDIC’s administrative control. The BIF insures deposits in most commercial banks and many savings banks and the money for it comes from the assessments contributed by member banks and also from investment income earned by the fund. On the other hand, SAIF, created for the thrift industry, succeeded the Federal Savings and Loan Insurance Corporation as the insurer of deposits to specified limits at savings associations and many savings banks (Pennachi G.G.( 2009)).

The primary purposes of the Deposit Insurance Fund (DIF) are:

- to insure the deposits and protect the depositors of insured banks and
- to resolve failed banks.

The DIF is funded mainly through quarterly assessments on insured banks, but also receives interest income on its securities (for more details see § 1.1.2.3.). The
FDIC’s operating expenses and loss provisions associated with failed banks minimize the size of the DIF. In 2010, the FDIC’s fund management authority was revised by the Dodd-Frank Act setting requirements for the Designated Reserve Ratio (DRR) - the DIF balance divided by estimated insured deposits. Moreover, the Dodd-Frank Wall Street Reform and Consumer Protection Act of 2010 redefined the assessment base, which is used by FDIC to calculate banks’ quarterly assessments (see below). In this way, a comprehensive, long-term management plan for the DIF was achieved by the FDIC, which not only could reduce pro-cyclicality and achieve moderate, steady assessment rates throughout economic and credit cycles but also could maintain a positive fund balance even during a banking crisis.

Figure 1.1: Simulation of fund loss and income data from 1950-2010

The FDIC's Board has to set a target or DRR for the DIF annually. Since 2010, the Board has adopted a 2.0 percent DRR each year according to a simulated analysis (Fig. 1.1) regarding the years 1950-2010 and the two major crises happened during these years. Actually, the FDIC believes that the 2.0% DRR is the minimum level needed to withstand future crises of the magnitude of past crises.

Assessments

It is widely recognized that a bank's capitalization is of utmost importance for it provides the main line of defense for absorbing unexpected losses, and therefore is an important predictor of default (Estrella et al., 2000). A measure of bank capitalization health is provided by Capital Ratio (CAR), which is defined as (Tier1+Tier2)/Risk-Weighted Assets. The FDIC has adopted a classification of capitalization based on a bucketing approach, which is different than the one that Basel has adopted. In
particular, according to the FDIC a bank is classified in one of the following buckets (Ellis, 2013):

1. **Critically Undercapitalized** if $CAR < 2%$
2. **Significantly Undercapitalized** if $2% \leq CAR < 6$
3. **Undercapitalized** if $6% \leq CAR < 8$
4. **Adequately Capitalized** if $8% \leq CAR < 10%$ and
5. **Well Capitalized** if $CAR \geq 10%$

This bucketing may seem ad hoc at first glance. However, at least for regulatory purposes, it is preferred to the simple dichotomous Basel classification that characterizes a bank either as undercapitalized or not, depending on whether its Capital Ratio falls below or above 8%. The intuition behind this bucketing is that not all undercapitalized banks should be treated as identical. For instance, according to the Basel rule a bank with a 3% CAR and a bank with a 7.5% CAR are both undercapitalized, but it is apparent that the former bank, ceteris paribus, is more vulnerable and more susceptible to default.

From the creation of the FDIC until 2010, a bank's assessment base was about equal to its total domestic deposits. A bank's assessment, determined each quarter, was calculated by multiplying its assessment rate by its assessment base. This was until April 2011, when the FDIC amended its regulations to define a bank's assessment base as its average consolidated total assets minus its average tangible equity in accordance with the Dodd-Frank Act.

The FDIC has chosen two different ways on determining the risk based assessment rates for small and large banks, namely, with less and more than $10 billion in assets, respectively. Thus, small banks are assigned by the FDIC in four risk categories based upon their capital levels and composite CAMELS ratings (Table 1.3). In general, Supervisory Group A comprises banks with CAMELS ratings of 1 or 2, Supervisory Group B with rating 3, and Supervisory Group C with rating 4 or 5.

**Table 1.3.:** Small Banks’ risk categories by the FDIC based upon their capital levels and composite CAMELS ratings

<table>
<thead>
<tr>
<th>Capital Groups</th>
<th>Supervisory Subgroups</th>
</tr>
</thead>
<tbody>
<tr>
<td>Well Capitalized</td>
<td></td>
</tr>
<tr>
<td>Adequately Capitalized</td>
<td></td>
</tr>
<tr>
<td>Under Capitalized</td>
<td></td>
</tr>
</tbody>
</table>

*Source: FDIC*
In Risk Category I, the assessment rate for each small bank is determined individually based on a combination of financial ratios and CAMELS component ratings. The maximum of these rates is also applied to the new banks which are insured for less than five years. In any other risk category, the banks are charged a single rate that depends on the risk category. For large banks, there are no risk categories, and they are assigned an individual rate based on a scorecard. The scorecard combines the following measures to produce a score that is converted to an assessment rate: CAMELS component ratings, financial measures used to measure a bank's ability to withstand asset-related and funding-related stress, and a measure of loss severity that estimates the relative magnitude of potential losses to the FDIC in the event of the bank's failure (Kuritzkes, A., Schuermann, T., Weiner S.M. (2002)).

In Table 1.4 the current assessment rates are provided. All rates are expressed as annual rates and are in basis points, which are cents per $100 of assessment base.

**Table 1.4.: The current assessment rates***

<table>
<thead>
<tr>
<th>Risk Category</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>Large Banks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Assessment Rate</td>
<td>5 to 9</td>
<td>14</td>
<td>23</td>
<td>35</td>
<td>5 to 35</td>
</tr>
<tr>
<td>Unsecured Debt Adjustment (added) **</td>
<td>-4.5 to 0</td>
<td>-5 to 0</td>
<td>-5 to 0</td>
<td>-5 to 0</td>
<td>-5 to 0</td>
</tr>
<tr>
<td>Brokered Deposit Adjustment (added)</td>
<td>N/A</td>
<td>0 to 10</td>
<td>0 to 10</td>
<td>0 to 10</td>
<td>0 to 10</td>
</tr>
<tr>
<td>Total Assessment Rate ***</td>
<td>2.5 to 9</td>
<td>9 to 24</td>
<td>18 to 33</td>
<td>30 to 45</td>
<td>2.5 to 45</td>
</tr>
</tbody>
</table>

* Source: FDIC; 
** The unsecured debt adjustment cannot exceed the lesser of 5 basis points or 50 percent of an insured depository bank's initial base assessment rate; 
*** Total assessment rates do not include the DIDA.

An annual rate is converted to a quarterly rate by dividing the annual rate by 4.

It is worth to mention that assessment rates for both large and small banks are subject to changes as follows:

- decrease for issuance of long-term unsecured debt, including senior unsecured debt and subordinated debt;
- increase for holdings of long-term unsecured or subordinated debt issued by other insured banks (the Depository Institution Debt Adjustment or DIDA); and
- for banks that are not well-rated or not well-capitalized, increase for significant holdings of brokered deposits.
1.2.3 Assuring the Solvency of FDIC

Funding deposit insurance

The experience that the FDIC gained all these years, from past crises, the management of the deposit insurance system, and the analysis of the funding requirements, reflects on the adopted system for funding itself. Nowadays, the FDIC has adopted a sophisticated system where risk is explicitly taken into account in determining the appropriate size of the insurance fund and what premiums banks pay.

In fact a well designed deposit insurance system has to satisfy some requirements, especially if it is considered that funding arrangements play a critical role in its success. These requirements are: i) to ensure that adequate funds are readily available to respond to problems as they arise and avoid delays in closing failed banks and ii) since it includes a risk-based pricing system can serve to minimize the moral hazard issue that too often accompanies even the most carefully designed insurance scheme (FDIC, (2014), www.fdic.gov)

Ex-ante funding

As it is mentioned before, the FDIC has two basic functions. The first one is the function of the regulator of Financial Institutions, whereas the second function is that of protecting the depositors against the risk (and the subsequent losses that could be realized) of failure, that stems from a defaulted bank.

Thus, the FDIC acts as an insurer as well. To achieve these goals, an ex-ante fund is utilized to ensure the rapid & efficient settlement of any claims by the depositors, should they arise from a defaulted bank. Ex-ante is Latin for the phrase “based on forecasts rather than actual results”, so the structure of an ex-ante fund is essentially based on the forecasts of the frequency and severity of future claims rather than the realized ones, which are made by depositors of a failed bank. On the contrary, ex-post assessments or alternative arrangements could have a detrimental effect on the time needed in order to settle the aforementioned claims, subsequently resulting in a possible loss of confidence to the system, which poses a severe threat for its existence.

An ex ante system is more rule-based and offers greater certainty than other systems – the funds are intended to be in place before they are needed. A question that
has arisen regards the optimum size of such a fund. The FDIC strives to maintain a fund, which is sufficient at all times to pay depositor claims. Simultaneously its aforementioned size should not have a negative effect (i.e. in terms of an increase) to the cost of capital for the financial institutions. The ultimate goal of the FDIC is to help to preserve the confidence in the banking system, which is paramount for the financial stability of the economy as a whole.

Until 1989, the premium rates were set by law and revenue could exceed expenses, so there was no target fund size. However, after a series of rolling recessions that resulted in a large number of bank failures; the Congress in 1989, instituted for the first time a target fund size in the form of a Designated Reserve Ratio, or DRR, equal to at least 1.25 percent of estimated insured deposits. But in 1996, this 1.25% DRR turns to be a hard target, as Congress prohibited the FDIC to charge anything the well-capitalized and well-managed banks when the fund was above that target. This resulted to a series of problems like:

- The formation of a vast number of financial institutions which accounted for almost 90% of the whole banking sector that were insured by the FDIC but did not pay any premium in exchange for that coverage.
- Given that the newly chartered banks, as well as the fast expanding and well-managed banks did not pay any premium, they diluted the DRR to the point that the index did not represent the actual fund size, thus posing a risk for its sufficiency to act as a preventive measure.
- As a result of the two aforementioned situations the premiums paid by the financial institutions came to be volatile, thus making it too difficult to maintain the fund size on the DRR target.

In 2006, just prior the recent crisis, Congress allowed the FDIC to manage the fund within a range of 1.15 and 1.50%. Nowadays, the FDIC, based not only on its experience but also to the best practices used by other financial institutions, developed a loss distribution model for estimating the optimal size for its fund.

More specifically, the deposit insurance is thought in a way of a portfolio comprising of credit risks with the exposure of each risk pertaining to different banks. Then a three step methodology is put into action. The first step comprises of the estimation of the probability of default, the loss given default, and the exposure that
each bank experienced at the time of its default. The goal of this step is to derive the expected loss for the bank. Continuing, an econometric model examines the relations (in statistical terms) of the aforementioned components, which contribute to the expected loss and purely economic variables (e.g. housing prices, interest rates etc.). The third and final step produces a distribution of future defaults and the losses that are caused by them to the deposit insurance fund. This is achieved by the simulation of a vast number of diverse scenarios that are derived by the use of an economic scenario generator (Kahn C.M., Santos, J.A.C, (2001)).

According to the aforementioned methodology and by simulating the historical and the expected data for the period 1950-2010, a 2% DRR is required for avoiding a crisis. In moving toward this goal, the DRR is required to reach the minimum requirement of 1.35 percent by 2020 from the present 0.63 percent. Thereafter, the FDIC’s plan is to systematically increase the fund toward the 2 percent target. Currently, the reserve ratio is only.

It is worth to note that, the 2 percent is taken as a soft target. This means that the FDIC plans to reduce rates to produce the long-term average rate when the reserve ratio reaches 1.15 percent. Once the reserve ratio reaches 2 percent, the plan provides for rates to be reduced gradually, but not to zero, as the reserve ratio grows.

The last resort for the FDIC to maintain a solvent deposit insurance fund is a mechanism called emergency funding. As it happens with any financial institution the FDIC, has the choice to use backup lines of credit. It goes without saying that the utilization of this emergency fund carries with it certain borrowing limitations and interest requirements. Concluding, it should be stressed out that the FDIC is employing it in order to overcome the risk of insolvency, caused by extreme financial situations.

In 2009 during the most recent crisis, the FDIC faced a situation where the banking industry had been beneficiaries of extraordinary government assistance, and the industry and public were suffering from what was termed bailout fatigue. In order to overcome such a situation and with industry support, the FDIC implemented a prepaid assessment requirement, an approach not previously considered, to boost the Deposit Insurance Fund’s liquidity. In fact, the FDIC required all banks to prepay an estimated three years of deposit insurance premiums. In this way, the FDIC succeeded
to provide important liquidity to the fund without requiring government support. and was a successful means of meeting emergency funding needs.

1.2.4. Failure Resolution Methods

In its 80 year history, the FDIC has accumulated a great deal of experiences pertaining to resolution and regulation of bank and thrift institutions. FDIC’s procedure for selecting a resolution method has evolved over the years but it is mainly based on three important rules: i) the most appropriate failure-resolution method should be the maintenance of public confidence and stability in the banking system ii) should encourage market discipline against risk-taking and iii) should be cost-effective. In determining the most appropriate resolution method, the FDIC primarily tries to minimize disruption to the community and minimize the government’s role in owning, financing, and managing financial institutions (Ellis, D., (2013)).

The FDIC, based on its experience, has chosen three basic resolution methods for failing institutions, namely, Purchase and Assumption Transactions, Deposit Payoffs, and Open Bank Assistance Transactions. Of course, these basic resolution methods have changed and refined over the years depending on the results they caused and the FDIC used different approaches to find the most efficient way of managing bank and thrift failures. In fact, the resolution process itself went through a series of changes and adjustments throughout the crisis period because of ever-changing market conditions and legislation that prompted innovative cooperation between the government and the private sector. Although it could be argued that the best methods through which a failed institution is resolved is the less expensive, the FDIC still possesses sufficient latitude to customize particular resolution methods within that framework.

In the following paragraphs these three basic resolution methods are described:

The purchase and assumption (P&A) transaction is a resolution transaction in which the acquirer, a healthy insured institution, purchases some or all of the assets of a failed banking institute or thrift and assumes, at a minimum, all insured deposits and may assume all of their deposit liabilities. Under circumstances, the FDIC, as insurer, assists the ‘assuming’ institute to complete the transaction. On the other side, the acquirer, as a part of the P&A transaction, usually pays a premium to the FDIC for the assumed deposits, which decreases the total resolution cost.
The P & A method was the favored resolution policy of the FDIC and the Resolution Trust Corporation (RTC) during the crisis. Actually, from 1980-1994, the FDIC handled 73.5% of the failing institutions with P&As; whereas 67.4% of the assets and 69.2% of the deposits were handled through P&A transactions. The RTC handled 66.5% of the institutions and 73% of the deposits were handled with P&As.

The deposit payoff method was introduced by the FDIC in 1983. It involves the transfer of insured deposits and secured liabilities of a failed bank to a healthy institution that agrees to act as the FDIC’s agent. This method saved the FDIC and the RTC overhead expense and provided an opportunity for the agent bank to attract new customers. Deposit payoffs were used when no acquirer could be found or if the FDIC or the RTC did not receive a less costly bid for a P&A transaction and between 1980 - 1994, deposit payoffs were used with 18.3% of the total. Insured deposit transfers (a form of deposit payoff) were used with 10.9% of the failed banks.

The open bank assistance (OBA) transaction, is a resolution method in which the FDIC provided an insured bank at risk of failure with financial help in the form of loans, contributions, deposits, asset purchases, or the assumption of liabilities. Generally, the FDIC required new management and called for a private sector capital infusion. OBA was used to facilitate the acquisition or merger of a failing bank or thrift by a healthy institution. Due to restrictions imposed under the Federal Deposit Insurance Corporation Improvement Act (FDICIA) of 1991 and under The Resolution Trust Corporation Completion Act of 1993, which amended the Federal Deposit Insurance Act of 1950, OBA is no longer a commonly used resolution method. The major criticism of OBA was that the shareholders of failing institution benefited from government assistance even though the FDIC required shareholders to dilute their interest. OBAs were used with 8.2% of the failed banks (Pennachi G.G. (2009)).

1.3. FORECASTING BANK DEFAULTS USING CAMEL FACTORS

1.3.1. Prediction Methods of Bank Failures

Bank supervisory agencies are responsible for monitoring the financial conditions of commercial banks and enforcing related legislation and regulatory policy. An issue of major importance in bank supervision is the timely assessment of
the status risk profile of every bank with respect to the default risk, which poses threat for the stability of the financial system as a whole. This is achieved by identifying all the risks that could contribute to a bank becoming troubled and subsequently failing (i.e. defaulting). In this section we will present a set of risk factors, which carry some useful information pertaining to the condition of a bank. The ultimate goal is to facilitate the regulator of a system in order to enforce a prudential regulation.

In fact, predicting the default risk for banks, loans and securities is a classic, yet contemporary issue. The pioneer work of Altman (1968), which suggested using the so-called “Z score” to predict firms’ default risk, is followed by hundreds of research articles on this issue (see the review articles: Kumar and Ravi (2007), Fethi and Pasiouras (2009) and Demyanyk and Hasan (2010)). The necessity on this research also comes from the repeated occurrence of banking crises during the past two decades such as the Asian, the Russian and the Brazilian bank crisis, as well as the most recent one of 2007 (subprime mortgage crisis), which evolved into a world financial crisis that is still ongoing. This kind of crises indicates that although the central banks have employed various early warning systems to monitor the risk of banks, safe guarding the banking system is not an easy task. Regulators in the United States must conduct on-site examinations of bank risk every 12–18 months according to the Federal Deposit Insurance Corporation Improvement Act of 1991 (FDICIA). In order to preserve the safety and soundness of banks, they use a rating system (known as the CAMELS rating).

In the literature there are a numerous of econometrics and operations research methods used to describe, predict and remedy financial crises and mortgage defaults but most of the time, each method is not by its own capable to accurate predict financial crises and defaults of banks. On the other hand, operations research combines mathematical modeling, statistics, and algorithms, whereas the intelligence techniques, like neutral networks (NN), are developed from the field of artificial intelligence and brain modeling. Too many studies have shown that intelligence modeling techniques can be applied for predicting the bank failures and crises. Thus, Alam et al. (2000) demonstrate that fuzzy clustering and self-organizing neural networks provide classification tools for identifying potentially failing banks; whereas Celik and Karatepe (2007) referred that artificial NN models can predict the rates of
nonperforming loans relative to total loans, capital relative to assets, profit relative to assets, and equity relative to assets.

Continuing, some of the statistical and intelligence techniques in their analysis of the banking crises will be discussed.

Among the statistical techniques analyzing and forecasting bank failures, discriminant analysis was the leading technique for many years (Karels and Prakash (1987), Haslem et al. (1992)). Discriminant analysis (DA) is a tool for analyzing cross-sectional data with three main subcategories: linear, multivariate, and quadratic. The main drawback of discriminant analysis is that it requires a normal distribution of regressors; if this is the case, generalized linear models, such as Logit regression models may be employed. Also instead of discriminant analysis models, hazard or duration analysis models can be used if one wants to analyze time series data on bank, firm, or loan defaults. The combination of discriminant analysis, generalized linear models, and principal component analysis can lead to an integrated early warning system, which has more predictive ability than the other models used in the literature. (Canbas et al. (2005)).

Evaluating statistical and intelligence techniques (Davis and Karim, 2008a), like the generalized linear models and the Signal Extraction Early Warning System methods it is found that the choice of estimation models makes a difference in terms of indicator performance and crisis prediction. More precisely, logit model performs better as a global early warning system and Signal Extraction is preferable as a country-specific one. Davis and Karim (2008b) also tested the logit and binominal tree approaches in predicting the recent subprime crisis in the US and UK. Although they found that among global early warning systems for the US and UK, the Logit performs the best, it is proven that this model, as many others, has only a small ability to predict the crises. West (1985) combined the Logit model with factor analysis, to measure and depict banks’ financial and operating characteristics and as a result he came up with important descriptive variables similar to those used for CAMELS ratings.

Among intelligence techniques, neural networks (NN) are the most widely used. The NN model contains mathematical and algorithmic elements that mimic the biological neural networks of the human nervous system. The structure of NN models changes, based upon external or internal information that flows through the network,
during the learning phase and uses nonlinear function approximation tools to test the relationship between explanatory factors.

Boyacioglu et al. (2008) utilized similar financial ratios as those used in CAMELS ratings, in order to compare the efficacy of various neutral networks, support vector machine and multivariate statistical methods, pertaining to the bank failure prediction problem in Turkey. They tested four different neutral network architectures, namely MultiLayer Perceptron (MLP), Cross-Layer (CL), Self-Organizing Maps (SOM) and Learning Vector Quantization (LVQ) and four multivariate statistical methods, namely multivariate discriminant analysis; K-means cluster analysis, and Logit regression analysis. According to their comparison, MLP and LVQ were the most successful models in predicting the financial failure of banks in Turkey. Besides the NN technique, other models have been developed like the back-propagation neural networks (BPNN) model, which is a multilayer NN model. The important feature of BPNN is that the errors generated by units of a hidden layer are calculated by back-propagating the errors of the output sent by levels of its corresponding layer. BPNN is one of the most commonly used methods for classification and prediction problems since it has indicated that it outperforms other models (Tam (1991), Tam and Kiang (1992), Bell (1997), Swicegood and Clark (2001)).

Utilities Additives DIScriminants (UTADIS) is a bank failure prediction model developed by Kosmidou and Zopounidis (2008). UTADIS is based on an additive value function which is performed through mathematical linear programming techniques instead of statistical methods. This permits UTADIS to perform better than other traditional multivariate data analysis techniques, since it is not sensitive to the statistical problems. Authors indicated that UTADIS is quite efficient for the evaluation of bank failure as early as four years before it occurs. Zopounidis and Doumpos (1999) also explore whether the UTADIS methods are applicable for analyzing business failure. They compare this method to DA and standard Logit and Probit statistical models.

The multi criteria decision aid (MCDA) method is a model that allows for the analysis of several preference criteria simultaneously. Doumpos and Zopounidis (2000) proposed the Multi-Group Hierarchical Discrimination (M.H.DIS) method to determine the risk classes to which the alternatives belong. According to the authors
this method performs better than traditional multiple discriminant analysis. MCDA can be used in credit ratings and bank soundness (for example, Gaganis et al. (2006)) as well as to replicate the credit rating of Fitch on Asian banks (Pasiouras et al (2007)). They found that MCDA is more efficient in this case by comparing its accuracy with that of discriminant analysis, and ordered Logit.

Tsionas and Papadakis (2009) provided a statistical framework, which can be used with stochastic data envelopment analysis. Data envelopment analysis is a nonparametric performance method used to measure the relative efficiencies of organizational or decision-making units. In order to make inference on the efficiency scores, the authors use a Bayesian approach to the problem, setup around simulation techniques. They also tested the new methods on the efficiency of Greek banks, and found that the majority of the Greek banks operate close to market best-practices.

1.3.2. Forecasting Defaults Using CAMEL Factors

Bank supervisory agencies are responsible for monitoring the financial conditions of commercial banks and enforcing related legislation and regulatory policy. A useful tool for accomplishing that task is the family of the so called CAMEL risk factors. The acronym “CAMEL” refers to the five components of a bank’s condition that are assessed: Capital adequacy, Asset quality, Management ability, Earnings, and Liquidity.

The ratios which are used are (Christopoulos et al., 2011):

- the Capital Adequacy Ratio, a very important index acting as a buffer for potential risks and also used for important decisions with regard to growth. In order a bank to have capital adequacy, this ratio should be higher than 8%, namely the total amount of capital must be over 8% of its risk-weighted assets and is given by the formula:

\[ CAR = \frac{(TIER \ I + TIER \ II)}{\text{Risk-Weighted Assets}} \]  \hspace{1cm} (1.1)

where TIER I forms the basic and own capital and includes: common and preferred stocks, the bank's minority rights in subsidiary companies, convertible bonds and TIER II: forms the bank's supplementary capital.

- the Asset Quality Ratio, which is based on evaluating credit risks associated with a bank's portfolios. It is determined as:
(Total Non-Performing Loans > 90 Days – Provisions) / Total Loans \hspace{1cm} (1.2)

The total of non-performing loans over 90 days has been defined by Basel II as a critical point for loan repayment. The bank’s accuracy of the provisions of these delays is better as lower is this ratio.

- the *Management Quality Ratio* equals to Management expenses over sales and forms the mechanism that makes the decisions to ensure the bank's smooth course of operation. The lower the ratio, the better for the bank, because it indicates that it has good management.

- the *Earnings Ratios*, this ratios reflect the bank's ability to absorb losses, expand its financing, and its ability to pay dividends to its shareholders, and helps develop an adequate amount of own capital. The two ratios Return On Assets (ROA) and Return On Equity (ROE) are equal to:

\[
ROA = \frac{\text{Net Profits}}{\text{Total Assets}} \hspace{1cm} (1.3)
\]
\[
ROE = \frac{\text{Net Profits}}{\text{Own Capital}} \hspace{1cm} (1.4)
\]

A value between 1% and 2.5% for ROA is satisfactory for the efficiency of a bank, whereas the higher the value of ROE, the more the bank can use its capital in a efficient way.

- The *Liquidity Ratios* are two different ratios, the loan to totals deposits and the circulating assets to total assets. The lower the former ratio is, the better the bank's liquidity status, while a value of less than one offers security for loans since deposits alone are sufficient to cover such loans. The second ratio offers banks the ability to know the extent of their liabilities that may be covered by its not directly available assets. The higher the bank’s ratio, the better its liquidity status. These two ratios are formulated as follows:

\[
\text{Loans to Total Deposits} = \frac{\text{Total Loans}}{\text{Total Deposits}} \hspace{1cm} (1.5)
\]
\[
\text{Circulating Assets to Total Assets} = \frac{\text{Circulating Assets}}{\text{Total Assets}} \hspace{1cm} (1.6)
\]

The official CAMEL ratings are commonly viewed as typical summary measures of banks’ overall financial condition. Ratings are assigned for each component on a scale from 1 to 5 in addition to the overall rating of a bank’s financial condition. Banks with ratings of 1 or 2, initiate few, if any, supervisory concerns, while banks with ratings of 3, 4, or 5 indicate moderate to extreme degrees of supervisory concern.
The general consensus in financial literature is that the information contained in CAMEL ratings can be efficiently exploited for the supervisory monitoring of banks. Since this information is usually disclosed to the financial markets (and therefore affects the prices of bank securities), the supervisory information in CAMELS ratings appears to be useful in the public monitoring of banks as well. Table 1.5 indicates areas, as well as significant ratios used for denoting strengths or weaknesses of a bank under CAMEL ratings.

TABLE 1.5. Ratios under CAMEL Ratings

<table>
<thead>
<tr>
<th>Aspect</th>
<th>Ratios</th>
<th>Formula</th>
<th>Symbol</th>
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</thead>
<tbody>
<tr>
<td>Capital Adequacy</td>
<td>Capital Adequacy Ratio</td>
<td>Capital &amp; Reverse / Total Risk Weighted Assets</td>
<td>C</td>
</tr>
<tr>
<td></td>
<td>Leverage Ratio</td>
<td>Debt / Total Shareholder’s Equity</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Return on Equity</td>
<td>Net Profit / Paid up Capital + free Reserves</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Net-worth Protection</td>
<td>Total equity / Non-performing Loan</td>
<td></td>
</tr>
<tr>
<td>Asset Quality</td>
<td>Percentage of Classified Loan</td>
<td>Non-performing Loan / Total Loan</td>
<td>A</td>
</tr>
<tr>
<td>Management Capacity</td>
<td>Income per Employee</td>
<td>Total Profit / Total Employees</td>
<td>M</td>
</tr>
<tr>
<td></td>
<td>Expenses per Employee</td>
<td>Total Cost / Total Employees</td>
<td></td>
</tr>
<tr>
<td>Earnings Ability</td>
<td>Net Investment Margin</td>
<td>Profit from Investment / Total Investment</td>
<td>E</td>
</tr>
<tr>
<td></td>
<td>Net Profit Margin</td>
<td>Profit after Tax / Total Loan &amp; Advance</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Diversification Ratio</td>
<td>Non-Interest Income / Total income</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Earning per Share</td>
<td>Profit after Tax / Total Number of Share</td>
<td></td>
</tr>
<tr>
<td>Liquidity</td>
<td>Loan to Deposit Ratio</td>
<td>Total Loan / Total Deposit</td>
<td>L</td>
</tr>
<tr>
<td></td>
<td>Liquid Assets to Total Deposit Ratios</td>
<td>Liquid Asset /Total Deposit</td>
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</tr>
<tr>
<td></td>
<td>Earning Assets to Deposit</td>
<td>Earning Asset / Total Deposit</td>
<td></td>
</tr>
</tbody>
</table>

With regard to the background of introducing CAMEL, it was originally adopted by the regulators of North American Commercial banks and it covers five aforementioned areas of performance. In the early 1970s; federal regulators of the US developed CAMEL rating system to appraise the performance of the Commercial banks. Later in 1979, the uniform financial institution’s rating system was adopted to provide federal regulatory agencies with a framework for rating financial condition and individual banks (Siems and Barr, 1998). Since then, the application of CAMEL has spread up dramatically with respect to assessing the financial strengths of one of
the basic constituents of money markets, i.e. commercial banks. In this connection, Piyu rightly observed: “Currently, financial ratios are often used to measure the overall soundness of a bank and the quality of bank management. Thus, bank regulators may use financial ratios to help evaluate a bank’s performance as part of CAMEL rating system” (Piyu, 1992).

In the new globalised financial system, the banks' economic situation changes more rapidly in relation to the past. As a result, supervisory authorities were led towards changing their way of approach and assessment, paying more emphasis on ways to overcome and manage risks (Doumpos & Zopounidis, 2009). Thus, a further area of assessment was added, that of the initial S, indicating market risk. This took place in 1995 by the US Federal Reserve (Fed) and the Comptroller of the Currency (Hafer, 2005), who replaced CAMEL with CAMELS and added a management assessment scale from 1 (optimum) to 5 (worse) for risk management (Broz, 1997).

An issue of major importance in bank supervision is which risk factors contain useful information for bank failure. With respect to predicting bank default, Barker and Holdsworth (1993) reported that CAMEL ratings are providing substantial information, about the condition and performance of banks. On the other hand, Cole and Gunther (1998) dealt with a similar question and found that, the information in supervisory CAMEL assessments deteriorates quickly over time. More specifically, analyzing data for the period 1988-1992, they found that new (less than 6 months old) CAMEL ratings more accurately predict bank financial distress than financial ratios can. However, financial ratios are better predictors than older (more than 6 months old) CAMEL ratings. O’Keefe and Dahl (1996) concluded that this result is asymmetric in the sense that CAMEL ratings become less reliable over time for banks with deteriorating finances, but not for banks with improving financial condition.

DeYoung et al., (1998) examined whether private supervisory information is useful in pricing the subordinated debt of large bank holding companies (BHCs). They use an econometric method to estimate the private information component of the CAMEL ratings for the BHCs’ lead banks and regressed it onto subordinated bond prices. They found that this approach adds significant explanatory power to the regression after controlling for publicly available financial information and that it is incorporated into bond prices about six months after an exam.
Berger and Davies (1998) used CAMEL ratings to examine the behavior of BHC stock prices in the eight-week period following an exam of its lead bank and found out that CAMEL downgrades reveal unfavorable private information about bank conditions to the stock market. Berger et al. (1998) extended this analysis to investigate whether the information about BHC conditions gathered by supervisors is different from that used by the financial markets. More specifically they applied Granger causality analysis to the leading and lagging relationships between exam ratings and the actions of bank stakeholders in financial markets for 184 bank holding companies between 1989 and 1992. Their main finding was that lagged movements in BOPEC ratings (the safety and soundness ratings for bank holding companies, whose acronym stands for Bank subsidiaries, Other nonbank subsidiaries, Parent company, Earnings, and Capital adequacy) explain 1.6% of the ‘additional’ variation in shareholder market variables (i.e., stock returns, changes in insider and institutional shareholdings), but explain 4.1% of the ‘additional’ variation in bond ratings.

Hirtle and Lopez (1999) scrutinized on the usefulness of past CAMEL ratings in assessing banks’ current conditions. Their main finding was that, the private supervisory information contained in past CAMEL ratings provides further insight into bank’s current conditions, as summarized by current CAMEL ratings. The authors used data for the period from 1989 to 1995 and concluded that, the supervisory information gathered during the last on-site exam remains useful with respect to the current condition of a bank for up to 1.5 to 3 years.

Dincera et al. (2012) used the CAMEL ratios to illuminate the effects of the global 2008 crisis on the performance of Turkish banking sector. In addition they were exploited to study the trends before and after the crisis in the Turkish economy. Kabir and Dey (2012) studied the comparative performance of two leading private sector commercial banks in Bangladesh, on the basis of the CAMEL rating system.

The overall conclusion drawn from the related academic literature is that the supervisory information, as summarized by CAMEL ratings, is quite helpful for the enforcement of an efficient prudential regulation in conjunction with the ability to setup a rigorous early warning system. Thus, the information that flow from the banks to the regulator of the system facilitate her in order to promote and maintain the stability and continuing operation of the financial system, which is vital for the economy as a whole.
CHAPTER 2. RISK MODELING

Risk management has become an important topic for financial institutions, regulators, nonfinancial corporations and asset managers. A central issue in modern risk management is the measurement of risk. One may give a general definition of risk for an organization as follows: *risk is any event or action that may adversely affect an organization to achieve its obligations and execute its strategies.*

In Financial Risk Management the notion of risk has a more explicit definition and is traditionally divided into several subcategories such as market risk, credit risk, operational risk, liquidity risk, legal risk etc. In the framework of the present thesis, only the first two categories of risks are of chief interest. The term market risk refers to risks due to interest rate fluctuations, foreign exchange rate changes, changing markets, market prices commodity price changes etc. Credit risk is the risk undertaken by a lender when a counterparty in a financial agreement can not fulfill his/her commitments.

The need to quantify risk arises in many different contexts. For example, a regulator wishes to measure the risk exposure of a financial institution (e.g. a bank) in order to determine the amount of capital the institution should hold as a buffer against unexpected losses. Likewise, the clearing house of an exchange has to address the problem of setting margin requirements for investors trading on that exchange. In another context, insurance premiums compensate an insurance company for bearing the risk of the insured claims; apparently, the size of this compensation can be viewed as a measure of the risk of the insured claims.

In order to introduce the necessary notation for describing the methods used for quantifying risks, let us consider a given portfolio, e.g. a collection of stocks, bonds and/or risky loans, or the overall position of a financial institution. If we denote by \( V(t) \) the value of the portfolio at time \( t \), the profit for the time interval \([t, t + \Delta t]\), will be given by \( L(t + \Delta t) - L(t) \); therefore we may define the loss, for the same interval, as

\[
L_{[t,t+\Delta t]} = -L(t + \Delta t) + L(t).
\]  

(1.7)

Apparently the values of \( L_{[t,t+\Delta t]} \) depend on both the time instance \( t \) and the time horizon \( \Delta t \). Typical values for \( \Delta t \) are one day, ten days (two weeks, 5 trading days...
per week), one month or one year. In market risk the time horizon is typically one day or ten days, while in credit risk where the portfolio may consist of loans, the time horizon is often one year.

When we are studying the loss for a fixed time horizon, we shall usually suppress the subscript \([t, t + \Delta t]\), and denote the loss by \(L\). From the statistical point of view, the loss \(L\), is a random variable and the computation of the risk of a portfolio, requires the probability distribution associated with \(L\). The parameters of the loss distribution need to be estimated using historical data. One can either try to model the loss distribution directly or to model the risk-factors or risk-factor changes over the time horizon of interest. We shall present first, in Sections 2.1 and 2.2, a brief review of risk measures and later on, in Section 2.3, we shall discuss the problem of modelling the whole loss distribution.

In the financial literature, there have been suggested several risk measures which are based on the distribution of \(L\). A measure of risk, quite popular in portfolio theory, is the standard deviation of the loss distribution, i.e. \(\sigma = \sqrt{\text{Var}[L]}\). However the use of the standard deviation as a measure of risk has the following disadvantages:

- If the loss distribution has very heavy tails (which is quite common in finance), the variance of \(L\), may not exist or may converge to infinity. Therefore the standard deviation cannot be defined or used in practice.
- Profits and losses have equal impact on the variance (and, consequently, on the standard deviation) of the loss distribution. Therefore the standard deviation, as a measure of risk, cannot discriminate between distributions with potentially large positive losses and potentially large negative losses (i.e. large profits). In fact, the standard deviation does not provide any information on how large potential losses may be.

Other measures of risk, related to variance, have been proposed as well, such as semi-deviation, which counts only deviations below a target value, lower partial moments, which apply to a wider range of utility functions etc. However we are not going to focus on them, since their use in finance is quite limited.

In order to overcome the aforementioned disadvantages of the standard deviation as a measure of risk, two alternative risk measures have been suggested,
which are by now widely used in financial risk management (see, e.g. Altzner et al. (1999)): Value-at-Risk (VaR) and Expected Shortfall (ES). In the next two sections, we provide an overview of the existing approaches to measure risk by these measures and discuss their strengths and weaknesses.

In closing the present section, we should like to stress that the definition of the portfolio loss implicitly assumes that the composition of the portfolio remains unaltered over the time horizon $\Delta t$. Although this assumption seems realistic when studying daily losses, it becomes increasingly problematic for longer time horizons. We note that, in the context of the Basel Accords, it is formally required that the calculations of market risk for banks be made under the assumption that the portfolio composition remains unchanged over a holding period 10.

2.1 VALUE AT RISK

Value at risk (VaR) is nowadays a standard tool in risk management for banks, regulatory agencies and other financial institutions. Its greatest advantage is that it summarizes risk in a single, conceptually easy-to-understand number. This is probably the main reason why VAR has become an essential tool for conveying trading risks to senior managers, directors, shareholders etc.

As mentioned in the monograph by Jorion (2007) “J.P. Morgan was one of the first banks to disclose its VAR. It revealed in its 1994 Annual Report that its trading VAR was an average of $15 million at the 95 percent level over 1 day. Based on this information, shareholders then can assess whether they are comfortable with this level of risk. Before such figures were released, shareholders had only a vague idea of the extent of trading activities assumed by the bank.”

Value at Risk is defined as the worst loss over a target horizon such that there is a prespecified probability (confidence level) that the actual loss will be smaller\(^1\). Common confidence levels are 95%, 99% and 99.9% depending on the application. The BIS (Bank of International Settlements) proposes, for market risk, to compute ten-day VaR with confidence level 99%.

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\(^1\) In terms of supervisory regulation, VaR can be viewed as the amount of equity capital which is needed so that the confidence level is the probability that insolvency will not occur.
As an illustration, let us consider a 99% confidence level, and assume a VaR limit of say 70,000 Euro has been imposed on a certain trader. Then a loss of more than 70,000 Euro is expected to occur only once in every hundred trading days (on average).

In probability theory nomenclature, VaR is a quantile of the loss distribution. Let us next give the mathematical definition of VaR. As stated in the previous section, we shall denote by \( L \) the loss of a portfolio over a fixed time horizon. The loss \( L \) is a random variable and follows a continuous distribution, with probability density function \( f(x) \), and respective cumulative distribution function \( F(x) \). Then, for a given confidence level \( 0 < a < 1 \), the associated \( VaR_a \) satisfies the following conditions (see Figure 2.1)

\[
\Pr(L \leq \text{VaR}) = a \iff \int_{-\infty}^{\text{VaR}} f(x) dx = a \iff F(\text{VaR}) = a.
\]  

(1.8)

Figure 2.1. Value-at-Risk for a loss distribution with probability density function \( f(x) \) (\( a=0.99 \))

If, instead of using the random variable \( L \) which describes the loss, we made use of the profit, say \( X \), of the portfolio over a time horizon, then we have \( L = -X \) and the condition \( \Pr(L \leq \text{VaR}) \) takes on the form

\[
\Pr(-X \leq \text{VaR}) = a \iff \\
\iff \Pr(X \geq -\text{VaR}) = a \iff \\
\iff \Pr(X \leq -\text{VaR}) = 1 - a \iff \Pr(X \leq -\text{VaR}) = p
\]

(1.9)
where \( p = 1 - a \) (see Figure 2.2). In view of (1.8) one may state that \( \text{VaR} \) is the \( a \)-
(lower) quantile of the loss distribution, while (1.9) states that \( -\text{VaR} \) is the \( p = 1 - a \) quan-
tile of the profit distribution.

Let us denote by \( \text{VaR}(L) \) the Value at Risk of a portfolio whose loss is
described by the random variable \( L \). It can be proved mathematically that, for any
positive number \( c \) and any number \( d \), the following identity holds true

\[
\text{VaR}(cL + d) = c\text{VaR}(L) + d
\]

(1.10)

Therefore, the risk measured in \( \text{VaR} \) for \( c \) shares of a portfolio is \( c \) times the risk
of one share of this portfolio. In addition, adding \( (d < 0) \) or withdrawing \( (d > 0) \) an
amount of money from the portfolio changes this risk by the same amount.

The traditional methods for evaluating \( \text{VaR} \) are the parametric and the non-
parametric one. For the first approach, one assumes that the loss distribution can be
assumed to belong to a parametric family, e.g. the normal (Gaussian) distribution or
any other continuous statistical distribution. This approach is called \textit{parametric}
because it involves estimation of parameters, such as mean and the standard deviation.
This method is simple, convenient and produces accurate estimates of \( \text{VAR} \), provided
that the fitted distribution to our data is appropriate. However this is not true in
practical applications, especially in finance.

If we assume that the loss distribution is Normal with mean \( \mu \) and variance \( \sigma^2 \)
, then the condition \( \Pr(L \leq \text{VaR}) = a \) can be restated as follows

\[
\Pr\left( \frac{L - \mu}{\sigma} \leq \frac{\text{VaR} - \mu}{\sigma} \right) = a
\]

(1.11)

and making use of the fact that the random variable \( Z = (L - \mu)/\sigma \) follows a
standard Normal distribution \( \mathcal{N}(0,1) \) we conclude that

\[
\Pr\left( \frac{L - \mu}{\sigma} \leq \frac{\text{VaR} - \mu}{\sigma} \right) = a \iff \Phi\left( \frac{\text{VaR} - \mu}{\sigma} \right) = \alpha
\]

(1.12)

where \( \Phi(.) \) denotes the cumulative the distribution function of a standard normal
random variable. Therefore \( \text{VaR} \) can be evaluated by the following formula

\[
\text{VaR} = \mu + \sigma \Phi^{-1}(\alpha)
\]

(1.13)
where $\Phi^{-1}(.)$ is the inverse function of $\Phi$ and its values can easily be obtained from tables of the standard Normal distribution $N(0,1)$. 

Apparently, if the loss distribution is a standard Normal distribution (with mean 0 and standard deviation 1), $VaR$ will be given by the formula

$$VaR = \Phi^{-1}(\alpha);$$

for a standard Normal $VaR$ at a 95% level we get $VaR = \Phi^{-1}(0.95) = 1.65$ (see Figure 2.3).

Analogous formulae could be derived for other parametric families of distributions; we shall further elucidate on that issue later on in Chapters 5 and 6 where new distribution models are introduced which are quite flexible in fitting financial data where the presence of skewness and asymmetry is quite common (therefore the Normal distribution is unable to provide an appropriate fit).

The non-parametric approach makes no assumption about the shape of the loss distribution. According to this approach, one uses the available data to estimate the empirical distribution function $\hat{F}(x)$ of $L$ and then calculates $VaR$ from the condition (see (1.8))

$$\hat{F}(VaR) = \alpha.$$ (1.14)

The main advantage of the nonparametric method is its being model-free and hence is model robust and avoids bias caused by using a miss-specified loss distribution.

The value of $VaR$ obtained by the non-parametric approach is usually referred to as “empirical $VaR$” (because the empirical distribution is exploited in its evaluation) while the value deduced by the parametric method is referred as “parametric $VaR$”

In recent years, changes are occurring in the financial markets at a very fast pace. Regulators have to respond very quickly by reexamining capital standards...
imposed on financial institutions (commercial banks, securities houses, insurance companies etc.). In the past, capital requirements set by the regulators, were simplistic and rigid and did not reflect appropriately the underlying economic risks of these financial institutions. Nowadays, regulators favor risk-based capital charges that better reflect the economic risks assumed. These new standards are chiefly based on value-at-risk methods. This is quite reasonable, since VaR is a measure of extreme unexpected loss at some confidence level and directly translates into a measure of buffer capital.

The regulation of commercial banks provides a useful example of evolving capital requirements. The Basel Capital Accord of 1988 provided the first step toward a "safe and sound" financial system. Although at the first place, the minimum capital requirements for commercial banks were set so as to guard them against credit risks, the expanding trading activities of banks, led to an amendment which incorporated market risks as well. Banks are now obliged to use their own VaR (internal) models as the basis for their required capital for market risk. Thus VaR is officially being promoted as a good risk management practice since it offers a tool to measure the soundness of a banks operation in terms of probability of insolvency.

2.2 EXPECTED SHORTFALL

Although Value-at-Risk has become a very popular risk measure among practitioners it has several limitations. It ignores statistical properties of the significant loss beyond the percentile point of interest, therefore it does not give any information about how bad losses may occur when things go wrong. An insight on this issue, could be offered if one looked at the size of the “average loss" given that the loss exceeds the, say 99%, Value-at-Risk.

As an illustration of the problem if VaR is used as a single risk measure without any supplementary information, consider a bank where a VaR limit (at confidence level 99%) of say 70,000 Euro is imposed on a certain trader. Then a loss of more than 70,000 Euro should occur on average only once in every hundred trading days. However, by the definition of VaR, there is no difference between small and very large violations of the 70,000 Euro limit. A loss of 75,000 Euro and a loss of 750,000 Euro have exactly the same contribution in the evaluation of VaR. Therefore, if VaR is
used as a criterion for risk-adjusted compensation, the trader has an incentive to run a strategy which would create an additional profit in most cases, but at the expense of a probability just below 1% of huge losses.

Another problem with VaR is it does not satisfy the sub-additivity property. This means that, if it is calculated for each unit within a bank, the sum of the VaR’s of each unit could be lower than the VaR for the whole bank! Since it practically means that risk could be reduced by running each unit separately, the use of VaR as a risk measure, contradicts the idea of diversification (which is very important in the framework of modern portfolio). Mathematically, the lack of sub-additivity in VaR, has as immediate consequence that portfolio optimization will not work appropriately, which in turn means that the internal allocation of capital cannot be carried out in a consistent and safe way.

In view of the above deficiencies VaR, a risk measure alternative to it has been suggested, which could in fact be used to supplement the info provided by VaR. This measure fulfils the property of sub-additivity and offers useful information on the size of the “average loss” given that the loss exceeds the VaR level.

If \( X \) is a random variable, then the Expected Shortfall (\( ES(x_0) \)) at a specific point \( x_0 \in (-\infty, \infty) \) is defined as the following conditional probability

\[
ES(x_0) = E(X|X < x_0)
\]  
(1.15)

In financial applications, the Expected Shortfall is related to loss distributions and the point \( x_0 \) at which the evaluation of it takes place is usually the value of VaR at a specific confidence level. So let us consider again a random variable \( L \) describing the loss of a portfolio and assume that is follows a continuous distribution, with probability density function \( f(x) \), and respective cumulative distribution function \( F(x) \). For a given confidence level \( 0 < a < 1 \), denote by VaR the Value at Risk associated to \( L \). Then the Expected Shortfall at confidence level \( a \), is defined as

\[
ES_a = E(L|L > \text{VaR})
\]  
(1.16)

and can be expressed in terms of the probability density function \( f(x) \) of \( L \), as follows:
\[ ES_a = E(L|L > VaR) = \frac{\int_{VaR}^1 xf(x)dx}{Pr[L > VaR]} = \frac{\int_{VaR}^1 xf(x)dx}{1 - Pr[L \leq VaR]} \] (1.17)

or equivalently (c.f. (1.8))

\[ ES_a = E(L|L > VaR) = \frac{\int_{VaR}^1 xf(x)dx}{1 - a}. \] (1.18)

In words, Expected Shortfall is defined as the mean of all losses which are greater or equal than \(VaR\), i.e. it accounts for the average loss in the worst \(100(1-a)\)% cases.

It can be proved that Expected Shortfall can be expressed in the following form

\[ ES_a = \frac{1}{1-a} \int_{a}^{1} VaR_{q} dq \] (1.19)

where the symbol \(VaR_{q}\) has been used to indicate the value at risk at confidence level \(q\). In view of the last formula, Expected Shortfall can be considered as conditional version of value-at-risk, where conditional \(VaR\) is defined as the average \(VaR\) for all confidence levels above \(p\).

If the loss distribution is the standard Normal distribution \(N(0,1)\), then one may prove that

\[ ES_a = \frac{1}{1-a} \phi(\Phi^{-1}(a)) \] (1.20)

while, for a loss \(L\) following a Normal distribution with mean \(\mu\) and variance \(\sigma^2\), one gets

\[ ES_a = \mu + \frac{\sigma}{1-a} \phi(\Phi^{-1}(a)) \] (1.21)

the symbols \(\phi\) and \(\Phi\) appearing in the above formulae denote the probability density function and the cumulative the distribution function respectively, of a standard Normal random variable.

### 2.3 Statistical Distribution Models for Financial Data

As made clear in Sections 2.1 and 2.2, the risk measures that are traditionally used in finance (Value-at-Risk and Expected Shortfall) are based on loss distributions. Most modern measures of the risk in a portfolio are statistical quantities describing the conditional or unconditional loss distribution of the portfolio over some predetermined horizon. Although quite convenient for practical reasons, it is
problematic to simply rely on any one particular statistic to summarize the risk contained in a distribution.

Detecting and studying the loss distribution as a whole gives an accurate picture of the risk in a portfolio and can facilitate in many aspects. For example

- if estimated properly, the loss distribution may accurately reflect netting and diversification effects.
- loss distributions can be compared to get better insight: for instance, it makes perfect sense to compare the loss distribution of a book of fixed-income instruments and of a portfolio of equity derivatives, at least if the time horizon same in both cases.
- loss distributions of correlated losses can be combined to carry out an efficient study of composite portfolios,
- loss distributions can be compared across time horizons,

Two major concerns when working with loss distributions may be considered very carefully. The first pertains to the fact that, the estimation of the loss distribution is carried out through statistical techniques that make use of past data. If the laws governing financial markets change, these past data are of limited use in predicting future risk.

The second is related to accuracy. Even in a stationary environment it is difficult to estimate the loss distribution accurately, particularly for large portfolios; unfortunately, many sophisticated financial risk-management systems, are making use of crude statistical models for the loss distribution that do not have the potential to describe efficiently the available data (e.g. it is quite popular to make lots of assumptions related to normality).

However, this discussion should not be seen as a recommendation to avoid using loss distributions. Rather, it calls for improvements in the way loss distributions are estimated and, of course, for additional research in enriching the distribution gallery by members that can be fruitfully used in practical applications of the risk-management field.

Given historical loss data a risk manager typically wants to estimate the probability of future large losses to assess the risk of holding a certain portfolio. Therefore, the development of adequate parametric distribution models capable of
describing financial assets is of utmost importance. It is widely known that the normality assumption for various asset returns is not supported by empirical evidence; the literature reports extensively that the distributions of several assets (e.g. stock returns, exchange rates etc.) are usually highly peaked and heavy tailed when compared with normal distributions. Therefore, it is important to develop theoretical distribution models explaining asymmetry and heavy tail phenomena.

Mittnik et al. (1998) suggested, for fitting asset returns, the use of a number of parametric distributions, and found that the asymmetric Weibull, Student-\(t\) and the asymmetric stable distributions provide the best fit according to various measures. For a more recent application of a generalized asymmetric student-\(t\)-distribution to financial econometrics see Zhu and Galbraith (2010). Cont (2001) mentioned that the precise form of the tail of financial returns’ distribution is difficult to determine, and argued that, in order for a parametric distribution model to reproduce the properties of the empirical distribution, it must have at least four parameters (a location parameter, a scale parameter, a parameter describing the decay of the tails and an asymmetry parameter).

In \(VaR\) applications, the choice of an appropriate distribution for the innovation process is an important issue as it directly affects the ‘quality’ of the estimation of the required quantiles. Since the Normality assumption, leads to very poor performance for the resulting model, a number of alternative distributional models have been suggested including a standardized version of the skewed Student distribution, generalized gamma and the Burr distribution (see Giot and Laurent (2003), Lunde(1999), Grammig and Maurer (2000)).

Extreme value theory provides the tools for an optimal use of the loss data to obtain accurate estimates of probabilities of large losses or, more generally, of extreme events. Extreme events are particularly frightening because although they are by definition rare, they may cause severe losses to a financial institution or insurance company. Under certain conditions, EVT methods can extrapolate information from the loss data to obtain meaningful estimates of the probability of rare events such as large losses. This, in turn means that, accurate estimates of Value-at-Risk and Expected Shortfall can be obtained.
The area of extreme value theory led to an abundance of distributional models, that can describe quite accurately extreme events. For a book length account on extreme value theory and related distributions that interested reader is referred to Kotz and Nagarajah (200) and Novak (2012).

Tolikas and Gettinby (2009), motivated by work related to environmental studies, focused on the Generalized Extreme Value, Generalized Pareto and Generalized Logistic distributions that have been proved particularly effective in describing the probability distributions of extremes in environmental studies (see, for example, Peel et al. (2001)). On analyzing the Singapore stock market over the period 1973 to 2005 they found that the popular Generalized Extreme Value distribution is not the best model for both the extreme minima and maxima daily returns, and that a fatter tailed model is needed; There empirical analysis showed that the Generalized Logistic provided a better fit to both the minima and maxima. This is an important finding since current applications of extreme value theory in finance focus mainly on the Generalized Extreme Value distribution; however, the probabilities of the extreme events provided by Generalized Extreme Value distribution may underestimate these extreme events.

From the foregone analysis it is clear that, despite the extensive research work on the field of generating distribution models capable of describing financial assets, there is a continuous demand for enriching the distribution gallery with more sophisticated and/or simple to implement models as well as for improvements on the techniques used to estimate the parameters of loss distributions. The second part of this thesis will focus on this subject.
CHAPTER 3. AIM AND CONTRIBUTION OF THE THESIS

Bank supervisory agencies are responsible for monitoring the financial conditions of commercial banks, enforcing related legislation and regulatory policy. As already stated in Section 1.1 the Basel Committee on Banking Supervision issued in January 1996 the ‘Market Risk Amendment to the Capital Accord’ according to which, besides the tools enforced by the regulator, banks should develop their own internal models as a basis for measuring their market risk capital requirements, subject to strict quantitative and qualitative standards. In addition, Pillar 2, probably the most important of the three pillars of the Revised Capital Framework (Basel II), calls for Supervisory Review Process and states that a bank must assess solvency versus risk profile and review of an institution’s capital adequacy and internal assessment process. Finally, Basel III is primarily related to the risks of a bank run; in order to mitigate a risk it is required by the bank to maintain differing levels of reserves for various forms of bank deposits.

As it is mentioned in Section 1.2, the FDIC promotes the safety and soundness of insured depository institutions and the U.S. financial system by identifying, monitoring, and addressing risks, which have an impact to the Deposit Insurance Funds. Nowadays, FDIC insures deposits as high as $9 trillion and aims at structuring its system to best achieve the goals of ensuring depositor confidence and financial stability. The FDIC’s operating expenses and loss provisions associated with failed banks may absorb a substantial part of the Deposit Insurance Funds. To avoid such unexpected events, a well-established assessment system should be built which will be capable of predicting bank defaults as well as the size of the loss when it occurs (loss given default-LGD). The repeated occurrence of banking crises during the past two decades indicates that, although the central banks and regulators/supervising authorities have employed various early warning systems to monitor the risk of banks, safe guarding the banking system is not an easy task.

Predicting the default risk for banks, loans and securities is a classic issue, however research in this field is still very active since many challenging problems are yet to be addressed. For the supervisory monitoring of banks, some quantitative methods applied on useful financial ratios (risk factors) that are often in use to measure the overall soundness of a bank’s operation, have to be practiced. As already
mentioned in Section 1.3, a useful tool for accomplishing the supervisory task in this framework is the family of the CAMEL risk factors. Since their introduction, in the early 1970s, the application of CAMEL factors has spread up dramatically in the financial risk management area.

According to the foregone analysis, three issues of major importance in bank supervision and monitoring the financial conditions of the banking sector are the following:

a. which risk factors contain useful information for bank failure?

b. how can one develop an early warning system that could give timely signs of an on-coming crisis?

c. How can one model the stochastic behavior of loss related to risky investments that directly affect the safe operation of the bank and determine a “fair” contribution to deposited in an insurance fund?

The present thesis contributes to the above three issues which apparently interact with each other to a great extent. It is divided in four separate Sections (Chapters), each examining a particular aspect of this interaction. Chapters 4 and 5 consist the first part of the thesis, which contributes to the bank supervision machinery while Chapters 6 and 7 deal with risk modelling pertaining to loss distributions useful in describing financial assets or returns.

Let us briefly discuss the contents of each chapter of the thesis.

Chapters 1 and 2 offer a brief overview of the wider areas where the outcomes of the present thesis belong in.

In Chapter 4 we conduct a comparison of the two past decades in terms of US banks' capitalization mobility and persistence. To achieve that, we adopt a Markov Chain setup, based on the FDIC discretized version of Capital Ratio, according to which each bank is classified in one of four available buckets (see Section 1.1.2). The empirical analysis is performed on a panel dataset whose cross section consists of all US commercial banks and its time series dimension of the period 1992-2009 (annual intervals). The novelty of our analysis is that it focuses on the trajectory of the cross sectional capital ratio distribution. Using a Markov Chain setup whose states correspond to the FDIC buckets, we estimate the transition probabilities between
capitalization buckets; including the default state. The benefit of this approach is that it allows the quantification of migration probabilities of banks across the capitalization spectrum, among which those related to the default state are of special interest. Moreover, the analysis permits estimating direction-specific migration probabilities, i.e. improvements vs. deteriorations of capitalization. This information is particularly useful both for market participants, but especially for the regulator. On the academic front we contribute to the literature, but instead of estimating the speed of adjustment, we aim at recovering a more fundamental property of the discrete capitalization process, namely its mobility (or equivalently its persistence). Taking advantage of the large period spanned by our dataset, we perform a comparison of the bank system mobility, between the 90's and the 00's. This approach allows tracing any similarities and/or differences effectively over the whole period in which the Basel requirements are in place.

Our empirical analysis carried out in Chapter 4 documents several substantial differences in the anatomy of the two periods. In particular, system mobility has shown a step increase in the 00's, and what is more important, mobility takes the form of increased probability of capitalization deterioration. Moreover, the vulnerability of Critically and Significantly Undercapitalized banks has markedly increased, as shown by the corresponding default transition probabilities.

The study presented in Chapter 4 has several policy relevant findings. Firstly, it offers (annual) capitalization bucket-specific transition probabilities towards all possible states for the periods of interest (the 90's and the 00's). This is useful for the regulator who can anticipate in future periods (with similar economic environments) the likely movements across the capitalization spectrum, with special reference to the default state. Secondly, it provides a comparison of capitalization movement experiences between the past two decades. Thirdly, it introduces a methodology that could be practiced in the future both for comparing specific periods as well as for highlighting possible developing trends, thereof getting an early warning of oncoming systemic crisis in the bank sector.

The purpose of Chapter 5 is to investigate which bank metrics exhibit ex ante long term forecasting power for bank failures. As already mentioned in Section 1.3, Barker and Holdsworth (1993) reported that, with respect to predicting bank default, CAMEL ratings are providing substantial information, about the condition and
performance of banks. On the other hand, Cole and Gunther (1998) found that, the information in supervisory CAMEL assessments deteriorates quickly over time. The overall conclusion drawn from the related academic literature is that the supervisory information, as summarized by CAMEL ratings, is quite helpful for the supervisory monitoring of bank conditions.

In the related literature, several econometric methods have been used to describe, predict and remedy financial crises and mortgage defaults (either at a bank level, or for the whole banking system of a country) but in most cases, each single method is not by its own capable to perform a very accurate prediction.

Discriminant analysis has been for many years the leading technique for analyzing and forecasting bank failures, Haslem et al. (1992). The main drawback of discriminant analysis is that it requires an underlying multivariate normal distribution for the quantities being analyzed, although in some cases one can work with DA despite the violation of this assumption. More recently, regression logit models have been used efficiently for analyzing time series data on bank firm, or loan defaults. As stated by Canbas et al. (2005), a combined use of discriminant analysis, logit models and principal component analysis may result in an integrated early warning system, which has high predictive capacity.

In Chapter 5, we use a panel dataset whose cross section consists of all US commercial banks insured by the FDIC and its time series dimension of the period 2000:q1-2011:q4 (quarter intervals). Using ex ante information on several key CAMEL factors, we explore how many quarters prior to failure, they may contain useful signals for the excess vulnerability of banks. The novelty of the empirical analysis is that it focuses on signals pertaining not to specific banks, but rather to groups of banks exposed to certain level of the risk factors.

After justifying graphically that, there are no substantial differences between (subsequently) failed banks and their non-failed peers up until about 20 quarters prior to failure, we scrutinize on the period starting from that point till the final bank failure. Using as cut-points the quartiles, we define four risk exposure classes for each CAMEL factor and conduct a preliminary exploratory analysis (by the aid of simple graphs and tables) based on the distribution of future failures in the 4 risk exposure classes. We exploit the Correspondence Analysis technique to investigate the
connection between bank failure and the categorical variables describing the four exposure levels of the CAMEL risk factors under a multivariate framework. We describe in some detail how a logistic regression model can be built and used for studying bank failures over specific time horizons. Focusing on 5 time horizons, we built different logit models for each risk factor and horizon, containing as explanatory variables the three higher risk exposure classes (the lowest risk exposure class is used as a reference group).

Our empirical findings indicate that, four to five years prior to failure three of the CAMEL risk factors contain significant forecasting power for future failures. Looking at a 3 year horizon before failure, we uncover that the set of significant predictors becomes larger. Finally, the empirical results across CAMEL metrics two years prior to failure indicate (as expected, given the proximity to failure) that most of the risk factors (10 out of 11) contain significant forecasting power. Combining the findings of the Correspondence Analysis approach and the logistic regression techniques, we substantiate that, according to the group of CAMEL factors that send out signals for bank failure, we may breaking up the bank life cycle (prior to failure) to 3 stages: 3-5 years prior to failure (Stage I), 3 years prior to failure (Stage II) and finally 1 year prior to failure (Stage III).

The suggested grouping of CAMEL factors is further analyzed by exploiting Discriminant Analysis techniques, and the results are reassuring the existence of the 3 stages. Moreover, using the developed logistic models, we present estimates for the probabilities of bank failure per exposure class for the significant CAMEL risk factors of each stage. Finally, combining the techniques of Principal Component Analysis and logistic regression, we develop a single index for failure forecasting in each of the three stages of bank failure lifecycle.

The results reported in Chapter 5, are particularly useful both for market participants, but especially for the regulator, when the need arises to quantify the excess vulnerability of specific groups of banks across different time horizons (with respect to predicting bank default).

The second part of the thesis (Chapters 6 and 7) addresses the problem of establishing new distribution models that can be profitably used for describing
financial data, such as foreign exchange rates, daily log-returns in stock markets, returns of a given portfolio etc.

As mentioned in Chapter 2, the development of adequate parametric distribution models capable of describing financial assets is of great importance. This need arises from the widely accepted fact that the normality assumption for various asset returns is not supported by empirical evidence; according to numerous publications in the financial literature, distributions of several assets (e.g. stock returns, exchange rates etc.) are usually highly peaked and heavy tailed when compared with normal distributions. Apparently, inappropriate use of a Normal model, while the data are fat-tailed, will lead to inaccurate estimates of quantities that are highly affected by the distribution tails, e.g. Value at Risk and Expected Shortfall.

Motivated by the above observations, several suggestions on using existing or new theoretical distribution models have been made in order to model the asymmetry and heavy tail phenomena that are usually present in financial applications. To mention a few a standardized version of the skewed Student distribution, an asymmetric Weibull and Student-t distribution, a generalized asymmetric Student-t model, a generalized gamma distribution and the Burr distribution have been proposed by several authors. The three types of extreme value distributions, and the respective unified model (Generalized Extreme Value distribution), have also been extensively used in financial risk modeling. However, the probabilities of the extreme events provided by Generalized Extreme Value distribution may underestimate the actual chances of observing the extreme event of interest, a fact that may result in catastrophic loss.

Taking into account that, despite the extensive research work on the field of modeling financial assets, it seems useful to enrich the distribution gallery with more sophisticated and/or simple to implement models, in the second part of this thesis we focused on this goal. More specifically, in Chapter 6 we introduce a generalized Logistic distribution which offers quite remarkable adaptability in real data arising in finance. The key point for the creation of our generalized model is the well known property of the Logistic distribution that the logit transformation of its cdf, i.e. \( \log(F(x))/(1-F(x)) \) is a linear function of \( x \). The generalization was achieved by considering a family of distributions having respective logit transformation of Polynomial type and of course it includes the classical Logistic distribution as a
special case. An important feature of the new family of distributions, which are called Polynomial–Logistic distributions, is that it accommodates a variety of models with a plethora of shapes (unimodal, bimodal, symmetric and non–symmetric). In the same chapter we study the characteristics of the Polynomial-Logistic family of distributions, and present results for its moments. Finally, we illustrate how it can be used to gain an accurate parametric model the Euro foreign exchange reference rates of 6 major currencies (Canadian Dollar, US Dollar, Australian Dollar, Swiss Franc, Japanese Yen, and UK Pound) for the period 1/4/1999-12/31/2011.

In Chapter 7, we present a further generalization of the model studied in Chapter 6, by introducing two additional parameters that can be used to control for the asymmetry and the decay of tails observed in empirical data. This distribution nests a variety of distribution models with all types of shapes (unimodal, bimodal, symmetric and non–symmetric) and includes as special cases the generalized logistic distribution which has been extensively studied in the literature. The new distribution, named Generalized Polynomial-Logistic distribution, can be used in several frameworks, including modeling of the extreme minima and maxima daily returns, developing distributional models for describing extremes in environmental studies, as appropriate model for the innovation process in ARCH, GARCH and stochastic volatility models for fitting asset returns etc. In Chapter 7 we also provide several properties pertinent to the generation of random variables following Generalized Polynomial-Logistic distribution, aging properties, as well as exact expressions for its moments and absolute moments. We discuss several methods for estimating its parameters and offer some formulae for evaluating the Value at Risk and Expected Shortfall metrics. Finally, we exploit this family of distributions to model the daily log-returns of eleven major international capital markets (United States, Canada, England, Switzerland, Germany, France, Hong Kong, Japan, India, Australia and Brazil) for the period 1/4/1999-12/31/2012 and the Euro foreign exchange daily returns for 9 major currencies (Canadian Dollar, US Dollar, Australian Dollar, Swiss Franc, Japanese Yen, UK Pound, Hong Kong Dollar, Indian Rupee and Brazil Real) for the same period.

Chapter 8 gives a summary of the results presented in the thesis and discusses issues that could be the subject of future research.
PART II. THE USA BANKING SECTOR: CAPITALIZATION SPECTRUM AND DEFAULT PREDICTION
CHAPTER 4. A MIGRATION APPROACH FOR USA BANKS' CAPITALIZATION

4.1. INTRODUCTION

It is widely recognized that a bank's capitalization is of utmost importance for it provides the main line of defense for absorbing unexpected losses, and therefore is an important predictor of default (Estrella et al., 2000). Since the early 90's the Basel agreement has put in place a capitalization threshold, relative to risk-weighted assets, above which banks are advised to operate. In particular, banks were expected to maintain a Capital Ratio (CAR hereafter), defined as Tier1+Tier2/Risk-Weighted Assets, of at least 8%. One may identify two strands of empirical academic literature regarding banks' capitalization. The first focuses on the determinants of banks' capital ratios decisions, while the second investigates the speed-of-adjustment of capital ratio - paying special attention to the time required until an undercapitalized bank satisfies the regulatory threshold (e.g. Demstez et al., 1996; Flannery and Sorescu, 1996; Goldberg and Hudgins, 2002; Lindquist, 2004; Berger et al., 2008; Jokipii and Milne, 2008).

The Federal Deposit Insurance Corporation (FDIC hereafter), the US banking sector regulator, has adopted a finer classification of capitalization based on a bucketing approach. In particular, according to the FDIC a bank is classified in one of the following buckets:

1. Critically Undercapitalized if \( \text{CAR} < 2\% \),
2. Significantly Undercapitalized if \( 2\% \leq \text{CAR} < 6\% \),
3. Undercapitalized if \( 6\% \leq \text{CAR} < 8\% \),
4. Adequately Capitalized if \( 8\% \leq \text{CAR} < 10\% \) and
5. Well Capitalized if \( \text{CAR} \geq 10\% \).

This bucketing may seem ad hoc at first glance. However, at least for regulatory purposes, it is preferred to the simple dichotomous Basel classification that characterizes a bank either as undercapitalized or not, depending on whether its Capital Ratio falls below or above 8%. The intuition behind this bucketing is that not all undercapitalized banks should be treated as identical. For instance, according to the Basel rule a bank with a 3% CAR and a bank with a 7.5% CAR are both
undercapitalized, but it is apparent that the former bank, ceteris paribus, is more vulnerable and more susceptible to default.

In the present chapter we conduct a comparison of the two past decades in terms of US banks' capitalization mobility and persistence, adopting a Markov Chain setup, based on the FDIC discretized version of Capital Ratio. Using a panel dataset whose cross section consists of all US commercial banks and its time series dimension of the period 1992-2009 (annual intervals), we document several substantial differences in the anatomy of the two periods. The novelty of the empirical analysis is that it focuses on the trajectory of the cross sectional capital ratio distribution. In particular, by adopting the FDIC bucketing approach, we rely on the discretized version of the capital ratio. Using a Markov Chain setup whose states correspond to the FDIC buckets, we estimate the transition probabilities between capitalization buckets; including the default state. The benefit of this approach is that it allows the quantification of migration probabilities of banks across the capitalization spectrum, among which those related to the default state are of special interest. Moreover, the analysis permits estimating direction-specific migration probabilities, i.e improvements vs. deteriorations of capitalization. This information is particularly useful both for market participants, but especially for the regulator. On the academic front we contribute to the literature, but instead of estimating the speed of adjustment, we aim at recovering a more fundamental property of the discrete capitalization process, namely its mobility (or equivalently its persistence). Moreover, we exploit the large period spanned by our dataset to conduct a comparison of system mobility, as well as other properties of the Markov Chain, between the 90's and the 00's. Therefore, we will trace any similarities and/or differences effectively over the whole period in which the Basel requirements are in place.

Let us briefly discuss our main empirical findings. Starting with the estimated transition probabilities, we find that for both sub-periods the Adequately Capitalized and Well Capitalized banks exhibit the lowest transition probabilities towards default, in comparison to banks in any other capitalization bucket. However, there are substantial differences between the two sub-periods. In particular, for the period 1992-2000 the Adequately Capitalized and Well Capitalized categories transited to default with 0.66% and 0.02% probability respectively. In contrast, for the period 2001-2009 the corresponding probabilities showed a step increase, with Adequately Capitalized
and Well Capitalized categories transiting to default with probabilities of 4.9% and 0.11% respectively. A similar increase of default probability transitions is observed for banks belonging to any of the undercapitalized categories. For instance, during the 1992-2000 period these probabilities were 5.49%, 29.93% and 20.45% for the Undercapitalized, Significantly Undercapitalized and Critically Undercapitalized buckets respectively, while for the period 2001-2009 the corresponding probabilities raised to 36.56%, 66.67% and 53.85%.

Looking at the mobility of the system, defined as the tendency to change capitalization buckets, based on a battery of mobility indices we document a substantial increase in the post 2000 era. What is more important is that this mobility takes the form of increased probability of capitalization deterioration. Moreover, during the post 2001 period these deteriorations are more likely to transcend one capitalization bucket. That is during the 00's in comparison to the 90's, the US banking sector has exhibited greater mobility, with capitalization deteriorations (including defaults) been more likely and more abrupt.

4.2. DATA ISSUES

The dataset consists of all FDIC-insured commercial banks in USA for the period 1992-2009, providing us a sample with a total of 183,739 bank-year observations. Let the indices $k,t$ denote the cross-sectional unit (bank) and time period (year) respectively. Then, exploiting the FDIC buckets, we may discretize the $\text{CAR}$ variable by defining a new variable $\text{CAP}_{k,t}$ as follows:

$$\text{CAP}_{k,t} = \begin{cases} 
1, & \text{if bank } k \text{ is well capitalized in year } t \\
2, & \text{if bank } k \text{ is adequately capitalized in year } t \\
3, & \text{if bank } k \text{ is undercapitalized in year } t \\
4, & \text{if bank } k \text{ is significantly undercapitalized in year } t \\
5, & \text{if bank } k \text{ is critically capitalized in year } t 
\end{cases}$$

(4.1)

Table 4.1 shows the distribution of banks across $\text{CAP}$ categories by year. The last two columns of the Table indicate the number of defaulted banks and the number of existing banks per year. Figures 4.1 and 4.2 depict the number of defaults and the respective percentage per year respectively. Apparently the majority of defaults (more than half of the total defaults) occurred during the first and last year of the period examined (1992 and 2009).
Table 4.1. Distribution of banks across capitalization bucket & year

<table>
<thead>
<tr>
<th>Year</th>
<th>Critically Undercapitalized</th>
<th>Significantly Undercapitalized</th>
<th>Undercapitalized</th>
<th>Adequately Capitalized</th>
<th>Well Capitalized</th>
<th># of Defaults</th>
<th># of banks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1992</td>
<td>0.78</td>
<td>0.66</td>
<td>0.67</td>
<td>3.41</td>
<td>93.20</td>
<td>180</td>
<td>13,822</td>
</tr>
<tr>
<td>1993</td>
<td>0.53</td>
<td>0.23</td>
<td>0.18</td>
<td>1.73</td>
<td>96.96</td>
<td>50</td>
<td>13,196</td>
</tr>
<tr>
<td>1994</td>
<td>0.06</td>
<td>0.11</td>
<td>0.22</td>
<td>1.54</td>
<td>97.96</td>
<td>15</td>
<td>12,529</td>
</tr>
<tr>
<td>1995</td>
<td>0.02</td>
<td>0.04</td>
<td>0.18</td>
<td>1.13</td>
<td>98.57</td>
<td>8</td>
<td>11,902</td>
</tr>
<tr>
<td>1996</td>
<td>0.01</td>
<td>0.08</td>
<td>0.11</td>
<td>1.48</td>
<td>98.26</td>
<td>6</td>
<td>11,394</td>
</tr>
<tr>
<td>1997</td>
<td>0.03</td>
<td>0.04</td>
<td>0.11</td>
<td>1.78</td>
<td>98.03</td>
<td>1</td>
<td>10,871</td>
</tr>
<tr>
<td>1998</td>
<td>0.01</td>
<td>0.09</td>
<td>0.21</td>
<td>1.91</td>
<td>97.75</td>
<td>3</td>
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</tr>
<tr>
<td>1999</td>
<td>0.02</td>
<td>0.04</td>
<td>0.09</td>
<td>2.15</td>
<td>97.63</td>
<td>8</td>
<td>10,184</td>
</tr>
<tr>
<td>2000</td>
<td>0.03</td>
<td>0.05</td>
<td>0.05</td>
<td>2.26</td>
<td>97.54</td>
<td>7</td>
<td>9,867</td>
</tr>
<tr>
<td>2001</td>
<td>0.04</td>
<td>0.04</td>
<td>0.16</td>
<td>2.22</td>
<td>97.50</td>
<td>4</td>
<td>9,581</td>
</tr>
<tr>
<td>2002</td>
<td>0.02</td>
<td>0.02</td>
<td>0.06</td>
<td>1.39</td>
<td>98.50</td>
<td>11</td>
<td>9,325</td>
</tr>
<tr>
<td>2003</td>
<td>0.02</td>
<td>0.03</td>
<td>0.03</td>
<td>0.99</td>
<td>98.92</td>
<td>3</td>
<td>9,154</td>
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<tr>
<td>2004</td>
<td>0.00</td>
<td>0.02</td>
<td>0.02</td>
<td>0.94</td>
<td>99.02</td>
<td>4</td>
<td>8,951</td>
</tr>
<tr>
<td>2005</td>
<td>0.00</td>
<td>0.02</td>
<td>0.06</td>
<td>0.76</td>
<td>99.16</td>
<td>0</td>
<td>8,809</td>
</tr>
<tr>
<td>2006</td>
<td>0.01</td>
<td>0.00</td>
<td>0.07</td>
<td>0.64</td>
<td>99.28</td>
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<td>8,657</td>
</tr>
<tr>
<td>2007</td>
<td>0.00</td>
<td>0.06</td>
<td>0.11</td>
<td>1.03</td>
<td>98.80</td>
<td>3</td>
<td>8,513</td>
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<tr>
<td>2008</td>
<td>0.10</td>
<td>0.37</td>
<td>0.74</td>
<td>2.28</td>
<td>96.51</td>
<td>25</td>
<td>8,288</td>
</tr>
<tr>
<td>2009</td>
<td>0.25</td>
<td>0.94</td>
<td>1.15</td>
<td>2.15</td>
<td>95.51</td>
<td>143</td>
<td>7,994</td>
</tr>
<tr>
<td>All years</td>
<td>0.11</td>
<td>0.16</td>
<td>0.23</td>
<td>1.66</td>
<td>97.73</td>
<td>471</td>
<td>183,739</td>
</tr>
</tbody>
</table>

Notes: (a) Numbers denote percentages, (b) The number of banks decreases each year due to M&A’s and bank defaults, (c) Sums may differ from 100 due to rounding errors.

Figure 4.1. Number of defaults per year in USA for the period 1992-2009
Figure 4.2. Percentage of defaults per year in USA for the period 1992-2009

The distribution of defaults per year in the period excluding 1992 and 2009 is more clear in Figures 4.3 and 4.4 where the two years with the extreme observations have been removed from the graphs.

[Figures 4.3 and 4.4 in Appendix]

Figure 4.5. Percentage of USA banks belonging to the Well Capitalized category per year for the period 1993-2008

Looking at the annual overall unconditional distributions across years, one may easily verify that the vast majority of banks belongs to the Well Capitalized category, followed by those belonging to the Adequately Capitalized category. This is clearly illustrated in Figures 4.5 and 4.6 which depict the percentages (per year) of USA
banks belonging to the Well Capitalized and Adequately Capitalized category respectively for the period 1993-2008.

From Table 4.1, one can immediately see that in the last two years of the sample (2008, 2009), which correspond to the ongoing severe crisis, there has been a notable increase in the percentage of undercapitalized banks (see Figures 4.7, 4.8 and 4.9). In particular, Critically Undercapitalized banks in 2009 showed a fivefold increase in comparison to 2002. Even more pronounced changes are found in the Significantly Undercapitalized and Undercapitalized banks, which in comparison to their 2002 magnitude they have increased by 47 and 57 times respectively.

[Figures 4.6 4.7, 4.8 and 4.9 in Appendix]

The annual overall unconditional distributions of the four capitalization categories are shown in Figure 4.10 (banks belonging to the Well Capitalized category have not pictured because the respective percentage is very high and the scaling in the y-axis prevents seeing clearly the distribution of the other four categories). In this figure one may clearly observe the fast increase of percentages in the last 3 categories (Undercapitalized, Significantly Undercapitalized and Critically Undercapitalized).

**Figure 4.10.** Percentage of USA banks belonging each capitalization category across years for the period 1993-2008

**Notes:** On the horizontal axis 1, 2, 3 and 4 denote adequately capitalized, Undercapitalized, Significantly Undercapitalized and Critically Undercapitalized categories respectively
4.3. EMPIRICAL METHODOLOGY

Let us briefly describe our empirical tactic. The thrust of our analysis will be the estimation of Markovian transition matrices in order to investigate the (im-)mobility properties of the cross-sectional capital ratio, with the ultimate aim to compare these properties across two different decades. However, before embarking on this analysis we will estimate a Dynamic Random Effects Ordered Probit model which controls for various bank fundamentals. We choose this multivariate time series setup as our springboard since we are dealing with relatively large time periods. Moreover, the rationale for such a model is that if the Markovian property is to be meaningful, i.e the current state to depend on the previous state, then one should find evidence in favor for a dynamic component in the Ordered Probit model. In other words, if the lagged capital ratio contains significant explanatory power for the current capital ratio, then we would have statistical evidence for estimating the transition matrices.

4.3.1 Preliminary Analysis

Since the variable under scrutiny is not continuous, but rather shows a discrete ordered response, it is reasonable to exploit a Random Effects Ordered Probit model (Wooldridge 2005). Let us assume that $CAP$ is described by a latent target variable model as follows:

\[
CAP^*_t = x_t^* \beta + c_t + e_t,
\]

where $x_t$ is a vector of covariates, $c_t$ is the unobserved heterogeneity, $e_t | x \sim N(0,1)$ and $\beta$ denotes a vector of constant parameters. Denoting by $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ the cut points (threshold parameters) we may restate $CAP_t$ as:

\[
CAP_t = \begin{cases} 
1, & \text{if } CAP^*_t \leq \alpha_1 \\
2, & \text{if } \alpha_1 < CAP^*_t \leq \alpha_2 \\
3, & \text{if } \alpha_2 < CAP^*_t \leq \alpha_3 \\
4, & \text{if } \alpha_3 < CAP^*_t \leq \alpha_4 \\
5, & \text{if } CAP^*_t > \alpha_4 
\end{cases}
\]

Following the methodology suggested by Berger et al. (2008) we assume the banks adjust their current $CAP$ relative to its previous level, at a fraction of last year's deviation from the latent desired level, namely:
where $\theta$ denotes the speed-of-adjustment parameter, $\text{CAP}_{k,t-1}$ is last year's capitalization status and $\text{CAP}_k^*$ is the latent desired level.

In view of (4.2) and (4.4) the following Dynamic Ordered Probit model for banks' observed choices arises:

$$\text{CAP}_k = (1-\theta) \cdot \text{CAP}_{k,t-1} + x_{k,t}^* \beta + c_k + e_{k,t}. \quad (4.5)$$

In the econometric terminology, the quantity $1-\theta$ captures any persistence in $\text{CAP}$. Manifestly, in the case of full adjustment ($\theta = 1$) there is no persistence, while partial adjustment ($\theta < 1$) would indicate the presence of some persistence.

The parameters estimation in model (4.5) involves several complications that need be addressed (e.g. unobserved heterogeneity cannot be eliminated with standard techniques such as differencing). To circumvent this problem, $c_k$ will be treated as a random effect giving rise to a *Dynamic Random Effects Ordered Probit*.

The literature has proposed several factors as potential determinants of bank capitalization capturing diverse associated incentives and costs. In short these factors reflect Size (e.g. Stolz and Wedow, 2005; Jokippi and Milne, 2008), Risk (e.g. Alfon et al., 2004; Lindquist, 2004), Profitability (e.g. Demsetz et al., 1996; Hellmann et al., 2000), Asset Quality and Market Discipline (e.g. Davenport and McDill, 2006; Flannery and Rangan, 2008).

We empirically proxy these factors with data obtained from FDIC's Statistics on Depository Institutions as follows$^1$:

**Size**: by the Logarithm of Total Assets; **Risk**: by the ratio of Risk Weighted Assets to Total Assets; **Profitability**: by the ratio of Net Operating Income to Total Assets; **Asset Quality**: by the ratio of Noncurrent Assets to Total Assets; **Market Discipline**: by the ratio of Uninsured Deposits to Total Deposits.

### 4.3.2 A Markov Model for Capital Ratio

Let us first present the basic terminology and techniques we will use from the theory of Markov models (see Noris, 1998; Ross, 1996). Consider a sequence of

---

$^1$ Bank-specific factors enter the model with one period lag in order to avoid endogeneity issues.
random vectors \( \{d_t\}_{t \geq 0} \) as denoting the distribution of banks across capitalizations buckets (states) each year, and the state-space \( \mathcal{S} = \{1, 2, \ldots, n\} \). The parameters of the Markov model \([\{d_t\}_{t \geq 0}, \mathcal{S}]\) are its transition probability matrix \( \mathbf{P} \) and its initial distribution \( \mathbf{\pi}_i, i \in \mathcal{S} \). The evolution of the distribution \( d_t \) over time is described as follows:

\[
d_t = \mathbf{P} \cdot d_{t-1}
\]

(4.6)

The typical element \( p_{ij} \) of the transition probability matrix \( \mathbf{P} \) indicates the transition probability that a bank is in state (capitalization bucket) \( j \) at time \( t \), given it was in state \( i \) in time \( t-1 \). By definition \( p_{ij} \geq 0, \forall i, j \) and since the set of states is exhaustive, the row elements of the matrix \( \mathbf{P} \) add up to unity: \( \sum_{j=1}^{n} p_{ij} = 1 \) for all \( i \).

Matrix \( \mathbf{P} \) summarizes all the \( n^2 \) transition probabilities that correspond to all possible movements across capitalization buckets. It is not unusual for a Markov chain to possess an absorbing state, in which once the chain enters, it is impossible to escape. In the case of banks this absorbing state is of crucial importance since it corresponds to default. Thus, the addition of the default state would reveal important migration properties such as the probability of defaulting within a year from any given current capitalization level. The absorbing state is typically treated as the last state in the state space \( \mathcal{S} = \{1, 2, \ldots, n\} \), in which case the transition probability matrix takes the form:

\[
\mathbf{P} = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1n-1} & p_{1n} \\
p_{21} & p_{22} & \cdots & p_{2n-1} & p_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\]

(4.7)

The estimation procedure for the entries of the transition probability matrix, using empirical data is achieved as follows.

Let \( N_{ij} = \sum_{t} 1(d_{t-1} = i, d_t = j) \) be the count of all transitions from state \( i \) to state \( j \) in the reference period, where \( 1(\alpha) \) is an indicator function that takes the value of unity when the argument \( \alpha \) is true, and zero otherwise. Let us also denote by \( N_{i} = \sum_{j \in \mathcal{S}} N_{ij} \) the number of transitions starting from state \( i \). Then the likelihood function, takes on the form:
\[ L = p_0 \prod_{i \in \mathbb{S}} \prod_{j \in \mathbb{S}} p_{ij}^{N_{ij}} = p_0 \prod_{i \in \mathbb{S}} L_i \left( p_{ij} \right) \]  

(4.8)

where \( L_i(p_{ij}) = \prod_{j \in \mathbb{S}} p_{ij}^{N_{ij}} \) depends on the elements of the \( i \)-th row of the transition probability matrix. The log-likelihood then becomes

\[
\ln L = \mathcal{L} = \sum_{i \in \mathbb{S}} \sum_{j \in \mathbb{S}} N_{ij} \ln p_{ij} \quad \text{s.t.} \quad \sum_{j \in \mathbb{S}} p_{ij} = \sum_{j=1}^{n} p_{ij} = 1, \quad p_{ij} \geq 0 
\]  

(4.9)

The likelihood maximization yields

\[
\hat{p}_{ij} = \frac{N_{ij}}{\sum_{j \in \mathbb{S}} N_{ij}} = \frac{N_{ij}}{\sum_{j=1}^{n} N_{ij}} = \frac{N_{ij}}{N_{\cdot \cdot}} 
\]

as the asymptotically unbiased and normally distributed ML estimator of \( p_{ij} \)

(Anderson and Goodman, 1957).

### 4.3.3 Capital ratio migrations

As it becomes apparent from the construction of a transition matrix, important information is encapsulated in its main diagonal, for it denotes the probability that the chain remains in the same state between two successive periods. In terms of Markov chain terminology, this property is usually referred as persistence. In the context of bank capitalization, persistence is perceived as a situation where a bank tends to remain in its current capitalization bucket. On the other hand, the off-diagonal elements of the transition matrix describe the ability of the Markov chain to shift in different states between successive periods. An interesting property, if satisfied, is that of Irreducibility, which implies that every state of the process is accessible by all other states (may be visited from any given state with a positive probability, not necessarily in one step).

The smaller the values of the off-diagonal elements of the transition matrix are, the lower mobility is expected between different states. In the extreme case where the off-diagonal elements are zero, we have the absolute (or complete) persistence case, which will be described by a transition probability matrix coinciding with the identity matrix \( \mathbf{I} \). In the case of absolute (or complete) persistence, the probabilities of remaining in the same bucket between any two consecutive periods would equal 1, and therefore no transitions in two successive periods (and consequently between any two periods) would be feasible.
4.3.4 (Im-)Mobility Assessment by the Aid of Mobility Indices

The assessment of the degree of mobility exhibited in an empirical transition matrix is based on indices that have been widely applied in the context of income mobility (Atkinson, 1970; Geweke, et al., 1986; Maasoumi and Zadvakili, 1986; Bigard et al., 1998; Schluter, 1998) and credit rating migration (Jafry and Schuermann, 2004; Trück and Rachev, 2006).

Let us first present briefly the main mobility indices used for investigating migrations in transition matrices. There are two classes of mobility indices that have been suggested in the relevant literature. The first class contains the so called **Summary Mobility Indices**, which describe the tendency of a bank to retain its capitalization bucket or to change capitalization buckets. The second class, called **Eigenvalue Based Indices**, contains indices which are expressed in terms of the eigenvalues of the empirical transition probability matrix under investigation and use all the information contained in the empirical transition probability matrix.

The Summary Mobility Indices (Bigard et al., 1998) are the **Immobility Ratio (IR)**, **Moving Up (MU)**, and **Moving Down (MD)**. Their main function is to assist us in conducting a preliminary analysis of the empirical transition matrix of interest, say $P = (p_{ij})$, and they are expressed as follows:

\[
IR = \left( \frac{\sum_{i=1}^{n} p_{ii}}{\sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}} \right) \times 100 = \left( \frac{\sum_{i=1}^{n} p_{ii}}{n} \right) \times 100 \tag{4.11}
\]

\[
MU = \left( \frac{\sum_{i<j} p_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}} \right) \times 100 = \left( \frac{\sum_{i<j} p_{ij}}{n} \right) \times 100 \tag{4.12}
\]

\[
MD = \left( \frac{\sum_{i>j} p_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}} \right) \times 100 = \left( \frac{\sum_{i>j} p_{ij}}{n} \right) \times 100 \tag{4.13}
\]

The immobility ratio index gives the percentage of the banks, whose bucket capitalization classification remained intact for the past year. In other words, the immobility ratio index captures any persistence in the system that would manifest itself in an increased probability of remaining to the current capitalization bucket. The other two indices reflect the percentage of the banks, which changed capitalization
bucket, moving to a higher/lower one within a year, and therefore pertain to the system’s mobility.

Let us now move to the second class of indices which are expressed in terms of the eigenvalues of the empirical transition matrix. The indices we employ are the Prais-Shorrocks Index (Prais, 1955), Sommers-Conlisk Index (Sommers and Conlisk, 1979), Shorrocks Index (Shorrocks, 1978), Half Life Index (Theil, 1972), which we denote as $M_{ps}, M_{sc}, M_s$ and $M_h$ respectively. Let $\lambda_i, i = 1, 2, ..., n$ denote the transition matrix eigenvalues in a descending order (in absolute value). Then the above mentioned indices are defined as follows:

$$M_{ps} = \frac{n - tr(P)}{n - 1} = \frac{n - \sum_{i=1}^{n} \lambda_i}{n - 1} \quad (4.14)$$

$$M_{sc} = 1 - |\lambda_2| \quad (4.15)$$

$$M_s = 1 - |\det(P)| = 1 - \left| \prod_{i=1}^{n} \lambda_i \right| \quad (4.16)$$

$$M_h = e^{-h}, \text{ where } h = \frac{-\log 2}{\log |\lambda_2|} \quad (4.17)$$

All indices, with the exception of the $M_h$, assume values in the $[0,1]$ interval, with 1 denoting the highest degree of mobility and 0 the lowest. The $M_h$ indicates the speed of convergence towards the equilibrium distribution $\Pi = \lim_{r \to \infty} P^r$. More precisely it indicates how long it takes for the system to cover half of the deviation from equilibrium, and as such the half-life indicator ranges between zero (in the case of perfect mobility) and infinity (in the case of perfect immobility).

Another index which is indirectly related to the eigenvalues of a matrix is the Singular Value Decomposition Index (SVD) (Jafry and Schuermann, 2004) which is given by the formula:

$$M_{SVD} = \frac{\sum_{i=1}^{n} \sqrt{\lambda_i^2}}{n} \quad (4.18)$$
where \( \lambda_i^+ \) denote the (positive) eigenvalues of the matrix \((P-I)'(P-I)\). The matrix \(P-I\) is usually referred to as mobility matrix. The square roots of the (non-negative) eigenvalues of the matrix \((P-I)'(P-I)\) are called singular values and therefore \(M_{SVD}\) is the average of the singular values of the transition probability matrix \(P\).

4.3.5 Comparison of Transition Probability Matrices by the Aid of Distance Metrics and Indices

The indices mentioned in the previous section can only be used to assess an individual transition matrix's mobility. However, in cases where one wishes to compare directly two transition matrices, so as to decide whether they describe similar or dissimilar behavior of the between-states transitions, they are not particularly useful. In the context of our study, it would be of special interest to compare the transition matrices \(P_{pre}\) (referring to the 1992-2000 period) and \(P_{post}\) (referring to the 2001-2009 period). Essentially we would like to compare the two sub-periods in terms of various facets regarding system mobility and persistence. In addition, one could compare the \(P_{pre}\) and \(P_{post}\) matrices to the identity matrix \(I\) in order to assess their proximity to complete persistence (perfect immobility).

To achieve these goals, one needs appropriate distance metrics whose value depict the closeness between two matrices, in the sense that large distance values would designate that the matrices are highly dissimilar. Three of the most commonly used distance metrics, which are based on cell-by-cell differences, are the Euclidean distance; \(D_{L_2}\) (Bangia et al., 2002), the absolute deviations distance; \(D_{L_1}\) (Israel et al., 2000), and the maximum distance; \(D_{L_{max}}\) (Trück, 2004). To offer the algebraic representations of these distance metrics let us consider two transition matrices \(P = [p_{ij}]_{n \times n}\), and \(Q = [q_{ij}]_{n \times n}\). Then, the formulae for the distance metrics are given as follows:

\[
D_{L_2}(P,Q) = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} (p_{ij} - q_{ij})^2}
\]  

\[
D_{L_1}(P,Q) = \sum_{i=1}^{n} \sum_{j=1}^{n} |p_{ij} - q_{ij}|
\]  

(4.19)  

(4.20)
The literature provides several variations and extensions of the above metrics (see Jackson and Murray, 2004). Most of them can be represented by the general formula:

\[
D_{\text{weight}}(P, Q) = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} | p_{ij} - q_{ij} | \right)^{1/p}
\]

(4.22)

with the parameters \( r \) and \( p \) varying from -1 to 1 and 1 to infinity respectively. For \( r \) less than 0, the elements \( p_{ij} \) cannot be zero, or the fraction will be undefined.

Apparently the distances \( D_{\text{le}}, D_{\text{ni}}, D_{\text{max}} \) correspond to the special case \( r = 0 \) and \( p = 2, 1 \) and infinity respectively. The special case \( r = 1 \) and \( p = 1 \) yields the so-called **weighted absolute difference (WAD)** (Lahr, 2001):

\[
D_{\text{WAD}}(P, Q) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} | p_{ij} - q_{ij} |
\]

(4.23)

Obviously, \( D_{\text{WAD}}(P, Q) \neq D_{\text{WAD}}(Q, P) \) and therefore the symmetry condition is not met. To correct this anomaly, one may use any of the following induced metrics:

\[
D_{\text{WAD}}^{\text{average}}(P, Q) = 0.5 \left( D_{\text{WAD}}(Q, P) + D_{\text{WAD}}(Q, P) \right)
\]

(4.24)

\[
D_{\text{WAD}}^{\text{max}}(P, Q) = \max \left( D_{\text{WAD}}(Q, P), D_{\text{WAD}}(Q, P) \right)
\]

(4.25)

Trück (2004) and Trück and Rachev (2006) introduced the following distance metric, which is based on the singular value decomposition indices of transition probability matrices:

\[
D_{\text{SVD}}(P, Q) = \left| M_{\text{SVD}}(P) - M_{\text{SVD}}(Q) \right|
\]

(4.26)

Another class of distances that has been coined by Trück (2004) and Trück and Rachev (2006), (2009) is the class of **risk adjusted difference indices**. To define them, they introduced the following differences (defined for every pair \( (i, j), i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, n \)):

\[
d_i(i, j) = (i - j) \left( p_{ij} - q_{ij} \right)
\]

(4.27)
\[ d_3(i, j) = (i - j) \text{sign}(p_{ij} - q_{ij})(p_{ij} - q_{ij})^2 \]  

and then introduced the distance between matrices \( P \) and \( Q \) by the following expressions:

\[
D_1(P, Q) = \sum_{i=1}^{n} \sum_{j=1}^{n} d_1(i, j) \quad (4.29)
\]

\[
D_3(P, Q) = \sum_{i=1}^{n} \sum_{j=1}^{n} d_3(i, j) \quad (4.30)
\]

\[
D_5(P, Q) = \sum_{i=1}^{n} \sum_{j=1}^{n-1} d_3(i, j) + \sum_{i=1}^{n} nd_3(i, n) \quad (4.31)
\]

\[
D_6(P, Q) = \sum_{i=1}^{n} \sum_{j=1}^{n-1} d_3(i, j) + \sum_{i=1}^{n} n^2 d_3(i, n) \quad (4.32)
\]

\[
D_7(P, Q) = \sum_{i=1}^{n} \sum_{j=1}^{n-1} d_1(i, j) + \sum_{i=1}^{n} nd_1(i, n) \quad (4.33)
\]

\[
D_8(P, Q) = \sum_{i=1}^{n} \sum_{j=1}^{n-1} d_1(i, j) + \sum_{i=1}^{n} n^2 d_1(i, n) \quad (4.34)
\]

Trück (2004) and Trück and Rachev (2006) also suggested two additional indices denoted by \( D_2, D_4 \) which are defined as follows:

\[
D_k(P, Q) = \sum_{i=1}^{n} \sum_{j=1}^{n} d_k(i, j), \text{ for } k = 2, 4 \quad (4.35)
\]

where

\[
D_2(i, j) = \frac{i - j}{p_{ij}}(p_{ij} - q_{ij}), \quad D_4(i, j) = \frac{i - j}{p_{ij}} \text{sign}(p_{ij} - q_{ij})(p_{ij} - q_{ij})^2. \quad (4.36)
\]

We have not used these differences in our study, since some entries of the matrix \( P_{post} \) are zero and therefore the quantities \( d_2(i, j) \) and \( d_4(i, j) \) tend to infinity.

The aforementioned risk adjusted difference indices were introduced in order to

a. detect the direction of the transition (DIR property),

b. differentiate transitions to the default state by attributing a higher weight to shifts towards the default state (TD property)

c. take into account the difference between ‘close’ and ‘far’ migrations by introducing a coefficient \( i - j \) to measure the distance between two states (Migration Deviation - MD property).
Note that \( D_i(P,Q) = -D_j(Q,P) \) for \( i = 1,3,5,6,7,8 \), which although seems quite puzzling for a distance metric, it is in fact a major advantage for them because their sign offers information about the direction of the transition; more specifically, if \( D_i(P,Q) < 0 \) then in matrix \( P \) we have more intense capitalization bucket migrations towards higher states (deterioration in our context) of the state-space \( \mathcal{S} = \{1,2,...,n\} \).

It is of interest to note that the indices \( D_5, D_6, D_7, D_8 \) treat the default state in a different way than the rest, by attributing a higher weight to the former. The aforementioned modification was suggested by Trück (2004) since among the possible migrations, the transition to the default state is usually the most influential, and therefore such migrations deserve a higher weight. As one can easily figure out from the definitions of \( D_i(P,Q) \), the difference between \( D_5, D_7 \) and \( D_6, D_8 \) is that the former indices use a weight \( n \) for the default state, whereas the latter exploit a different multiplier (\( n^2 \)). Additionally, \( D_5, D_6 \) use squared differences between the transition probabilities while \( D_7, D_8 \) sum up the actual differences.

Trück (2004), in the context of credit rating, showed that some of the difference indices \( D_i(P,Q) \) are highly correlated to changes in Value-at-Risk figures due to shifts in probability mass of transition matrices. Later on, Trück and Rachev (2006) performed a detailed study on a 20-year history of Moody’s migration matrices and verified that the correlation of indices \( D_i(P,Q), i = 5,6,7,8 \) to Value-at-Risk exceeds 90%. This fact substantiates their usefulness for comparing capitalization bucket migrations between different time periods, which is the main aim of this study.

4.4 EMPIRICAL RESULTS

4.4.1 Ordered Probit Estimation Results

Table 4.2 summarizes the estimation results from the Dynamic Random Effects Ordered Probit model controlling for bank-specific factors that are assumed to affect capital ratio movements. The parameters of main interest are the coefficients of the lagged capital ratio, which in both sub-periods turn out to be positive and highly statistically significant. These findings indicate that, on average, a bank’s capital ratio
is strongly influenced by its capitalization in the previous period. Therefore we have the required statistical evidence to proceed to the (Markov) transition models.

Table 4.2. Ordered Probit Regression Results

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Lagged CAP</td>
<td>1.0946***</td>
<td>1.0300***</td>
</tr>
<tr>
<td></td>
<td>(0.0214)</td>
<td>(0.0308)</td>
</tr>
<tr>
<td>SIZE</td>
<td>0.0083</td>
<td>-0.0581***</td>
</tr>
<tr>
<td></td>
<td>(0.0081)</td>
<td>(0.0081)</td>
</tr>
<tr>
<td>RISK</td>
<td>-0.0137***</td>
<td>-0.0119***</td>
</tr>
<tr>
<td></td>
<td>(0.0005)</td>
<td>(0.0005)</td>
</tr>
<tr>
<td>PROFITABILITY</td>
<td>0.03378***</td>
<td>0.0524***</td>
</tr>
<tr>
<td></td>
<td>(0.0057)</td>
<td>(0.0042)</td>
</tr>
<tr>
<td>ASSET QUALITY</td>
<td>-0.0358***</td>
<td>-0.1493***</td>
</tr>
<tr>
<td></td>
<td>(0.0038)</td>
<td>(0.0043)</td>
</tr>
<tr>
<td>MARKET DISCIPLINE</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>(0.0000)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>Cut_1</td>
<td>0.3280</td>
<td>0.0147</td>
</tr>
<tr>
<td></td>
<td>(0.1503)</td>
<td>(0.1866)</td>
</tr>
<tr>
<td>Cut_2</td>
<td>0.8351</td>
<td>0.1219</td>
</tr>
<tr>
<td></td>
<td>(0.1480)</td>
<td>(0.1863)</td>
</tr>
<tr>
<td>Cut_3</td>
<td>1.1764</td>
<td>0.3921</td>
</tr>
<tr>
<td></td>
<td>(0.1475)</td>
<td>(0.1858)</td>
</tr>
<tr>
<td>Cut_4</td>
<td>1.4948</td>
<td>0.6626</td>
</tr>
<tr>
<td></td>
<td>(0.1475)</td>
<td>(0.1854)</td>
</tr>
<tr>
<td>Cut_5</td>
<td>2.4442</td>
<td>1.2607</td>
</tr>
<tr>
<td></td>
<td>(0.1486)</td>
<td>(0.1856)</td>
</tr>
</tbody>
</table>

Diagnostics

|                         | -8074.1226     | -7531.7830     |
| Log-likelihood          |                |                |
| Likelihood Ratio test   | 5568.6400      | 4254.3600      |
| Observations            | 89078          | 78263          |

Notes: (a) Statistical significance at: ***1 percent levels (b) Numbers in brackets denote standard errors

4.4.2 Estimation of Sample Transition Matrices

As mentioned earlier, our main objective is the comparison of the capitalization bucket migrations for the periods 1992-2000 and 2001-2009. Therefore we shall be needing the empirical transition probability matrices referring to those two periods. Using the data of the period 1992-2000 we estimated the transition probabilities between all 6 states, thereof obtaining the empirical transition probability matrix $P_{pre}$ shown in Table 4.3. Note also that, according to the coding adopted an increase in the value of the variable $CAP_{k,t}$ designates a deterioration of the capitalization bucket.
whereas a decrease of its value corresponds to improvement of the capitalization bucket.

Table 4.3. Transition Probability Matrix for the period 1992-2000 (matrix $P^j_{\text{pre}}\,^{(a)}\,(b)$)

<table>
<thead>
<tr>
<th>From / To</th>
<th>Well Capitalized</th>
<th>Adequately Capitalized</th>
<th>Undercapitalized</th>
<th>Significantly Undercapitalized</th>
<th>Critically Undercapitalized</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>Well Capitalized</td>
<td>0.9882</td>
<td>0.0106</td>
<td>0.0007</td>
<td>0.0002</td>
<td>0.0001</td>
<td>0.0002</td>
</tr>
<tr>
<td>Adequately Capitalized</td>
<td>0.6049</td>
<td>0.3411</td>
<td>0.0282</td>
<td>0.0144</td>
<td>0.0048</td>
<td>0.0066</td>
</tr>
<tr>
<td>Undercapitalized</td>
<td>0.5055</td>
<td>0.2582</td>
<td>0.0659</td>
<td>0.0989</td>
<td>0.0166</td>
<td>0.0549</td>
</tr>
<tr>
<td>Significantly Undercapitalized</td>
<td>0.3139</td>
<td>0.1241</td>
<td>0.0949</td>
<td>0.1241</td>
<td>0.0437</td>
<td>0.2993</td>
</tr>
<tr>
<td>Critically Undercapitalized</td>
<td>0.1023</td>
<td>0.0114</td>
<td>0.0341</td>
<td>0.0113</td>
<td>0.6364</td>
<td>0.2045</td>
</tr>
<tr>
<td>Default</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Notes: (a) Numbers denote percentages (b) Sums do not add up to 1 due to rounding errors

Before calculating the mobility indices, it would be useful to gain some insights about movements between CAP categories by observing the respective sample transition probabilities. A graphical illustration for the elements appearing in Table 4.3 are given in Figure 4.11 which displays the transition probabilities from each capitalization category (denoted in the figure as 1, 2, 3, 4, 5) to the default state or any other capitalization category as shown in the last line in the horizontal axis.

Let us start our discussion by looking at the transition probabilities to the default state, as depicted in Figure 4.12. As expected, banks belonging to any of the undercapitalized categories run the highest default risk within a year (5.49%, 29.93% and 20.45% for the Undercapitalized, Significantly Undercapitalized and Critically Undercapitalized buckets respectively). What is interesting is that the Significantly Undercapitalized category has the highest probability of default (about 30%), while one would expect the Critically Undercapitalized group to exhibit the highest default probability. The Adequately Capitalized and Well Capitalized categories have very low probabilities of default, estimated at 0.66% and 0.02% respectively.
Figure 4.11. Transition probabilities (x100) for USA banks belonging to each capitalization category (time period 1992-2000)

Notes: On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively.

Figure 4.12. Transition probabilities (x100) to the default state for USA banks belonging to each capitalization category (time period 1992-2000)

Notes: On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively.
A simple inspection of Table 4.3 or Figure 4.11 reveals that persistence is substantially stronger at both ends of the capitalization spectrum, namely for the states describing the Well Capitalized and Critically Undercapitalized banks. In particular, Well Capitalized banks tend to remain in the same category almost with certainty (with probability of 98.82%), while Critically Undercapitalized banks retain their capitalization bucket with probability of about 64%. The other category with notable, but much lower, persistence is that of Adequately Capitalized banks which tend to retain their status with a probability just above 34%.

Regarding the other off-diagonal elements there are two findings that clearly stand out. First, it is noteworthy that there is a considerable fraction of banks completing transitions between rather distant states, and especially transitions to the Well Capitalized category. For instance, about 10% of Critically Capitalized banks, about 31% of Significantly Undercapitalized and about 50% of Undercapitalized banks transit to the Well Capitalized category. Second, banks which belong to the Undercapitalized group have an extremely high probability (roughly 88%) of transiting to another capitalization bucket. More specifically they face a probability of about 76% to improve their capitalization status (about 26% towards the Adequately Capitalized group, and about 50% towards the Well Capitalized group) while the probability of deteriorating arises to almost 12% (about 10% to the Significantly Undercapitalized category, and about 2% to the Critically Undercapitalized category).

Working in a similar fashion with the 2001 to 2009 part of the dataset we computed the empirical transition probability matrix $P_{post}$, shown in Table 4.4. This matrix gives a similar picture for the movements between $CAP$ categories in the period 2001-2009, albeit the transitions to lower capitalization buckets are now much more intense than the ones observed in $P_{pre}$.

A graphical illustration for the elements appearing in Table 4.4 are given in Figure 4.13 which displays the transition probabilities for the period 2001-2009 from each capitalization category (denoted in the figure as 1, 2, 3, 4, 5) to the default state or any other capitalization category as shown in the last line in the horizontal axis.
Table 4.4. Transition Probability Matrix for the period 2001-2009 (matrix $P_{post}$\(^{(a),(b)}\))

<table>
<thead>
<tr>
<th>From / To</th>
<th>Well Capitalized</th>
<th>Adequately Capitalized</th>
<th>Undercapitalized</th>
<th>Significantly Undercapitalized</th>
<th>Critically Undercapitalized</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>Well Capitalized</td>
<td>0.9851</td>
<td>0.0102</td>
<td>0.0011</td>
<td>0.0011</td>
<td>0.0004</td>
<td>0.0011</td>
</tr>
<tr>
<td>Adequately Capitalized</td>
<td>0.6795</td>
<td>0.1993</td>
<td>0.0291</td>
<td>0.0291</td>
<td>0.0035</td>
<td>0.0490</td>
</tr>
<tr>
<td>Undercapitalized</td>
<td>0.2581</td>
<td>0.1075</td>
<td>0.1720</td>
<td>0.1720</td>
<td>0.0430</td>
<td>0.3656</td>
</tr>
<tr>
<td>Significantly Undercapitalized</td>
<td>0.3333</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.6667</td>
</tr>
<tr>
<td>Critically Undercapitalized</td>
<td>0.3846</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.5385</td>
</tr>
<tr>
<td>Default</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Notes: (a) Numbers denote percentages (b) Sums do not add up to 1 due to rounding errors

Figure 4.13. Transition probabilities (x100) for USA banks belonging to each capitalization category (time period 2001-2009)

Notes: On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively

As depicted in Table 4.4 (see also Figure 4.14), banks belonging to any of the undercapitalized categories exhibit again the highest default risk within a year (36.56%, 66.67% and 53.85% for the Undercapitalized, Significantly Undercapitalized and Critically Undercapitalized buckets respectively), which are much higher than the respective risks estimated for the period 1992-2000.
The rather counter-intuitive fact of not observing the highest probability of default in the Critically Undercapitalized group is still present in this period (Significantly Undercapitalized category has instead, the highest probability of default). Adequately Capitalized and Well Capitalized categories have probabilities of default estimated at 4.9% and 0.11% respectively.

Persistence is now observed only on the upper end of the capitalization spectrum, with Well Capitalized banks tending to remain in this category almost with certainty (98.51%), while Critically and Significantly Undercapitalized banks have zero probability to retain their capitalization bucket. The other category with noteworthy persistence is that of Adequately Capitalized banks which tend to retain their status with a probability of almost 20%.

**Figure 4.14.** Transition probabilities (x100) to the default state for USA banks belonging to each capitalization category (time period 2001-2009)

Notes: On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively

As in the period 1992-2000, so in the 2001-2009 period there is a considerable fraction of banks completing transitions between rather distant states, and especially transitions to the Well Capitalized category. In particular, each year about 38% of Critically Capitalized banks, 33% of Significantly Undercapitalized and 26% of Undercapitalized banks transit to the Well Capitalized category. Also, it is of interest to note that banks which belong to the Undercapitalized group have an extremely high probability (roughly 58%) of transiting to another capitalization bucket. More specifically they face a probability of about 37% to improve their capitalization status.
(about 11% towards the Adequately Capitalized group, and about 26% towards the Well Capitalized group) and a probability of almost 22% of deteriorating to the Significantly or Critically Undercapitalized category.

A substantial difference between matrices $P_{pre}$ and $P_{post}$ is that in the latter there exist some zero entries (besides the ones appearing in the absorbing state row) which means that some states are not accessible by other states in one step, i.e. between successive years. For example the Adequately Capitalized are not accessible neither from the Significantly Undercapitalized nor from the Critically Undercapitalized category and the same applies for the Critically Undercapitalized and Significantly Undercapitalized categories which do not communicate in any of the two directions.

Figures 4.15-4.20 can be used for performing a graphical comparison of the transition probabilities to the default state (and the 5 capitalization buckets) between the two periods examined (1992-2000 versus 2001-2009)

[Figures 4.15 – 4.20 in Appendix]

From Table 4.15 it is clear that in the second period (2001-2009) a substantial increase has occurred in the transition probabilities to the default state for all the banks, with the banks belonging to Undercapitalized, Significantly Undercapitalized and Critically Undercapitalized categories exhibiting very high probabilities of default. Analogous findings can be traced from the rest figures, which depict the transition probabilities to the five capitalization categories in each of the two periods.

4.4.3 Calculation of Mobility Indices

Using the empirical transition matrices $P_{pre}$ and $P_{post}$ we calculated the values of the three summary mobility indices, the eigenvalue based indices as well as the SVD mobility Index, which are reported in Table 4.5.

Let us start the discussion of the results with the summary mobility indices. It is clear that the MU ratio is greater for the transition probability matrix concerning the period 2001-2009 as compared to the one for the period 1992-2000. This implies that after 2001 there were more transitions towards lower capitalization buckets as compared to the pre 2001 period. On the other hand, the MD ratio is rather similar between the two transition matrices. Therefore, based on the summary mobility
indices, we may conclude that mobility has substantially increased in the post 2001 era, with the majority of transitions involving movements towards deterioration of capitalization.

<table>
<thead>
<tr>
<th>Table 4.5. Mobility indices for transition probability matrices</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Index</strong></td>
</tr>
<tr>
<td>Immobility Ratio</td>
</tr>
<tr>
<td>Moving Up (deterioration)</td>
</tr>
<tr>
<td>Moving Down (improvement)</td>
</tr>
</tbody>
</table>

**Panel II: Eigenvalue Based Mobility Indices**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Prais-Shorrocks($M_{PS}$)</td>
<td>0.56886</td>
<td>0.75236</td>
</tr>
<tr>
<td>Sommers-Conlisk($M_{SC}$)</td>
<td>0.0007007766</td>
<td>0.0043276</td>
</tr>
<tr>
<td>Shorrocks($M_{S}$)</td>
<td>0.999568</td>
<td>1</td>
</tr>
<tr>
<td>Half Life ($h$)</td>
<td>988.781</td>
<td>159.822</td>
</tr>
</tbody>
</table>

**Panel III: Singular Value Based Indices**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mobility Index($M_{SVD}$)</td>
<td>0.567265</td>
<td>0.752142</td>
</tr>
</tbody>
</table>

The same conclusions are drawn, should one resort to the eigenvalue-based mobility indices. Recall that all eigenvalue-based mobility indices $M_{PS}$, $M_{SC}$ and $M_{S}$ take on values in $[0, 1]$ interval; yet, some of them may take either very large values ($M_{S}$) or very small values ($M_{SC}$) and therefore our conclusions should be drawn by comparing their values for the two periods we are interested in rather than using as a mobility evidence their closeness to 1. The Prais-Shorrocks index ($M_{PS}$), reveals a medium level of mobility (56.9%) for the 1992-2000 transition matrix. For the period 2001-2009 the same index shows a much greater mobility (75.2%). The singular value based index $M_{SVD}$ whose values are very close to the ones attained by the Prais-Shorrocks index leads to exactly the same conclusions. The $M_{SC}$ index paints the same picture as the $M_{PS}$ mobility index, if we take into account the fact that its values for the period 2001-2009 is much higher than the one exhibited for the period 1992-2000 (the respective ratio is about 0.0043276/0.0007007766=6.2).

Needless to say, the eigenvalue and singular value based indices, cannot differentiate between alternative configurations in the off-diagonal elements of the transition probability matrices and consequently do not provide any information with respect to the distribution of probabilities across neighboring and distant states.
The estimated half-life measure $M_h$ shows that it has decreased substantially from 988 for the period 1992-2000 to 159 for the period 2001-2009. **This finding reveals that for the post 2000 era there is considerably higher mobility than in the pre 2000.** To confirm this, note that looking at the transition probability matrices of order $2h$ (which is almost 320 for the period 2001-2009 and 1980 for the post 2001 period) one can easily verify that in the matrix $P^{320}_{\text{post}}$ the probabilities of default (no matter which the initial state was) exceed 75%, and the same holds true when we consider the matrix $P^{1980}_{\text{pre}}$.

### 4.4.4 Calculation of Distance Metrics and Indices

We continue the discussion of the results with the distances between the transition matrices $P_{\text{post}}$, $P_{\text{pre}}$ and the complete persistence (identity) matrix $I$. Table 4.6 reports the values of the most commonly used distance metrics when comparing each of the matrices $P_{\text{pre}}$ and $P_{\text{post}}$ to the complete persistence matrix $I$.

<table>
<thead>
<tr>
<th>Table 4.6. Distances between the transition probability matrices</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel I: Distances between $P_{\text{pre}}$, $P_{\text{post}}$ and $I$</strong></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>Cell by Cell Based Distances</strong></td>
</tr>
<tr>
<td>$D_{L_2}$</td>
</tr>
<tr>
<td>$D_{L_1}$</td>
</tr>
<tr>
<td>$D_{\text{max}}$</td>
</tr>
<tr>
<td><strong>Difference Distances</strong></td>
</tr>
<tr>
<td>$D_{\text{average}}$</td>
</tr>
<tr>
<td>$D_{\text{max}}$</td>
</tr>
<tr>
<td><strong>Singular Value Based Distance</strong></td>
</tr>
<tr>
<td>$D_{\text{SVD}}$</td>
</tr>
<tr>
<td><strong>Panel II: Risk Adjusted Difference Indices $D(P_{\text{post}}, P_{\text{pre}})$</strong></td>
</tr>
<tr>
<td>$D_1$</td>
</tr>
<tr>
<td>$D_2$</td>
</tr>
<tr>
<td>$D_3$</td>
</tr>
<tr>
<td>$D_4$</td>
</tr>
<tr>
<td>$D_5$</td>
</tr>
<tr>
<td>$D_6$</td>
</tr>
</tbody>
</table>
With regards to the “cell-by-cell based” and “Singular values-based” metrics we observe that the distances between $P_{post}$ and $I$ are greater than the respective distances between $P_{pre}$ and $I$. Taking into account the fact that $I$ describes perfect immobility, we derive the conclusion that there is higher mobility in the capitalization buckets in the period 2001-2009 as compared to the period 1992-2000. In addition this fact is corroborated by comparing the difference distance metrics $D_{WAD}^{average}\left(P_{pre},I\right)$, $D_{WAD}^{average}\left(P_{post},I\right)$, $D_{WAD}^{max}\left(P_{pre},I\right)$, $D_{WAD}^{max}\left(P_{post},I\right)$ who also portray the same picture as the cell by cell distances and the singular value based distances.

In order to arrive at a direct comparison between $P_{pre}$ and $P_{post}$ we shall use the values of the Risk Adjusted Difference Distances reported in Panel II of Table 4.6. With respect to the $D_1$ and the $D_3$ indices, from their negative sign, it is evident that we have a negative outlook, i.e. the probability of deterioration of the capitalization bucket for the typical bank is larger for the period 2001-2009. This means that after 2001 one expects to observe more deteriorations in terms of capitalization. Furthermore, based on these indices it is derived that there is a better (less negative) outlook, i.e. the probability of improvement of the capitalization bucket for a certain bank is bigger for the period 1992-2000. It is worth stressing that the Risk Adjusted Difference Distances $D_1$ and $D_3$ take into account the magnitude of transition (deviation of the migration). Therefore, the observed negative values of $D_1$ and $D_3$ indicate that in the years following 2001 it is more (less) likely to have deteriorations (improvements), that transcend one capitalization bucket.

We will complete this section by presenting the findings of the risk adjusted difference indices $D_5, D_6, D_7, D_8$. As mentioned in Section 4.3.2, indices $D_5-D_8$ assign different (amplified) weight to transitions to the default state of a migration matrix than the weights used for the transitions to the rest cells of it. The highly negative values of $D_5, D_6, D_7, D_8$ (especially in the cases of the last three) for the period 2001-2009 signify that transitions to the default state were more likely during that period.
Apparently, the conclusions based on distance measures, especially the ones derived through the risk adjusted difference indices, are in full compliance with the findings couched on the classical mobility indices.

4.5 FOCUSING ON THE CRISES PERIODS

As a further analysis and to ensure a direct comparability of the two sub-periods we repeat the previous analysis isolating now the crises periods. Essentially, we re-estimate the transition matrices as well as some of the mobility metrics for the sub-periods (1992-1994) and (2008-2009).

Panel A of Table 4.7 shows the empirical transition probability matrix for the 90’s crisis (see also Figure 4.21). As depicted in the last column of Panel A of Table 4.7 (see also Figure 4.23), during the 90’s crisis, banks belonging to any of the undercapitalized categories run the highest default risk within a year (6.86%, 29.90% and 18.42%) for the Undercapitalized, Significantly Undercapitalized and Critically Undercapitalized buckets respectively.

**Figure 4.21** Transition probabilities (x100) for USA banks belonging to each capitalization category (time period 1992-1994)

[Diagram showing transition probabilities]

**Notes:** On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively

What is interesting is that the Significantly Undercapitalized category has the highest probability of default (almost 30%), although one would expect the Critically
Undercapitalized group to exhibit the highest default probability. The Adequately Capitalized and Well Capitalized categories have very low probabilities of default, estimated at 0.75% and 0.03% respectively.

Table 4.7. Focus on Crises periods
Panel A. Transition Probability Matrix for the for the 90' crisis period (1992-1994\(^{(a)}\),\(^{(b)}\))

<table>
<thead>
<tr>
<th>From / To</th>
<th>Critically Undercapitalized</th>
<th>Significantly Undercapitalized</th>
<th>Undercapitalized</th>
<th>Adequately Capitalized</th>
<th>Well Capitalized</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>Critically Undercapitalized</td>
<td>72.37</td>
<td>0</td>
<td>2.63</td>
<td>1.32</td>
<td>5.26</td>
<td>18.42</td>
</tr>
<tr>
<td>Significantly Undercapitalized</td>
<td>6.19</td>
<td>14.43</td>
<td>8.25</td>
<td>15.46</td>
<td>25.77</td>
<td>29.90</td>
</tr>
<tr>
<td>Undercapitalized</td>
<td>1.96</td>
<td>10.78</td>
<td>7.84</td>
<td>28.43</td>
<td>44.13</td>
<td>6.86</td>
</tr>
<tr>
<td>Adequately Capitalized</td>
<td>0.60</td>
<td>2.08</td>
<td>2.53</td>
<td>29.66</td>
<td>64.38</td>
<td>0.75</td>
</tr>
<tr>
<td>Well Capitalized</td>
<td>0.02</td>
<td>0.02</td>
<td>0.06</td>
<td>0.72</td>
<td>99.15</td>
<td>0.03</td>
</tr>
<tr>
<td>Default</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Panel B. Transition Probability Matrix for the for the 00' crisis period (2008-2009)

<table>
<thead>
<tr>
<th>From / To</th>
<th>Critically Undercapitalized</th>
<th>Significantly Undercapitalized</th>
<th>Undercapitalized</th>
<th>Adequately Capitalized</th>
<th>Well Capitalized</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>Critically Undercapitalized</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>Significantly Undercapitalized</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>19.35</td>
<td>80.65</td>
</tr>
<tr>
<td>Undercapitalized</td>
<td>6.79</td>
<td>22.03</td>
<td>6.78</td>
<td>3.39</td>
<td>8.47</td>
<td>52.54</td>
</tr>
<tr>
<td>Adequately Capitalized</td>
<td>0.57</td>
<td>10.17</td>
<td>10.17</td>
<td>15.25</td>
<td>45.76</td>
<td>18.08</td>
</tr>
<tr>
<td>Well Capitalized</td>
<td>0.18</td>
<td>0.55</td>
<td>0.89</td>
<td>1.83</td>
<td>95.94</td>
<td>0.61</td>
</tr>
<tr>
<td>Default</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Notes: (a) Numbers denote percentages (b) Sums do not add up to 100 due to rounding errors

Persistence is substantially stronger at both ends of the capitalization spectrum. In particular, Well Capitalized banks tend to remain in the same category almost with certainty (with probability of 99.15%), while Critically Undercapitalized banks retain their capitalization bucket with probability 72.37%. The other category with notable, but much lower, persistence is that of Adequately Capitalized banks which tend to retain their status with a probability of about 30%.

Regarding the off-diagonal elements, it is noteworthy that there is a considerable fraction of banks completing transitions between rather distant states,
and especially transitions to the Well Capitalized category. For instance, just above 5% of Critically Capitalized banks, about 26% of Significantly Undercapitalized and about 44% of Undercapitalized banks transit to the Well Capitalized category. On the other hand, banks which belong to the Undercapitalized group have an extremely high probability (higher than 85%) of transiting to another capitalization bucket. More specifically they face a probability of about 73% to improve their capitalization status (about 28% towards the Adequately Capitalized group, and about 44% towards the Well Capitalized group) while the probability of deteriorating, without defaulting, arises to almost 13% (about 11% to the Significantly Undercapitalized category, and about 2% to the Critically Undercapitalized category).

Working in a similar fashion with the 2008 to 2009 part of the dataset we computed the empirical transition probability matrix for the 00’s crisis, shown in Panel B of Table 4.7 (see also Figure 4.22). This table displays a similar picture for the movements between CAP categories as the period 1992-1994 table, albeit the transitions to lower capitalization buckets are now much more intense.

**Figure 4.22** Transition probabilities (x100) for USA banks belonging to each capitalization category (time period 2008-2009)

**Notes:** On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively
Banks belonging to any of the undercapitalized categories exhibit again the highest default risk within a year (52.54%, 80.65% and 100% for the Undercapitalized, Significantly Undercapitalized and Critically Undercapitalized buckets respectively), which are much higher than the respective risks estimated for the period 1992-1994.

**Figure 4.23** Transition probabilities (x100) to default for USA banks belonging to each capitalization category

![Transition probabilities graph](image)

**Notes:** On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively.

The probabilities of default for Adequately Capitalized and Well Capitalized categories were estimated as 18.08% and 0.61% respectively. This is clearly displayed in Table 4.23 where a comparison to the corresponding transition probabilities for the 90’s crisis can be performed.

Persistence is now observed only on the upper end of the capitalization spectrum, with Well Capitalized banks tending to remain in this category almost with certainty (95.94%). It is notable that Critically and Significantly Undercapitalized banks have zero probability to retain their capitalization bucket. The other category with noteworthy persistence is that of Adequately Capitalized banks which tend to retain their status with a probability of about 15.25%.
Figures 4.24-4.28 can be used for performing a graphical comparison of the transition probabilities to the default state (and the 5 capitalization buckets) between the crisis periods examined (1992-1994 versus 2008-2009).

[Figures 4.24 – 4.28 in Appendix]

From Table 4.7 and Figure 4.23 it is clear that in the second period (2008-2009) a substantial increase has occurred in the transition probabilities to the default state for all the banks, with the banks belonging to Undercapitalized, Significantly Undercapitalized and Critically Undercapitalized categories exhibiting extremely high probabilities of default. Analogous findings can be traced from Table 4.7 and Figures 4.24-4.28 the rest figures, which depict the transition probabilities to the five capitalization categories in each of the two crisis periods. For example, one may easily figure out that, as in the period 1992-1994, so in the 2008-2009 period there is a considerable fraction of banks completing transitions between rather distant states, and especially transitions to the Well Capitalized category. In particular, each year more than 19% of Significantly Undercapitalized and about 8.5% of Undercapitalized banks transit to the Well Capitalized category. It is also of interest to note that banks which belong to the Undercapitalized group have a considerable probability (about 41%) of transiting to another capitalization bucket. They face a probability of about 12% to improve their capitalization status (just above 3% towards the Adequately Capitalized group, and about 8.5% towards the Well Capitalized group) and a probability of almost 29% of deteriorating to the Significantly or Critically Undercapitalized category.

A substantial difference between the matrices that encapsulate the 90’s and the 00’s crises is that in the latter there exist some zero entries (apart from the ones appearing in the absorbing state row) suggesting that some states are not accessible by other states in one step, i.e. between successive years. For example the Adequately Capitalized is not accessible neither from the Significantly Undercapitalized nor from the Critically Undercapitalized category and the same applies for the Critically Undercapitalized and Significantly Undercapitalized categories which do not communicate in any of the two directions.

Let us next comment on the results pertaining to mobility metrics given in Table 4.8. It is clear that the MU ratio is greater for the transition probability matrix
concerning the period 2008-2009 as compared to the one for the period 1992-1994. This implies that after 2008 there were more transitions towards lower capitalization buckets as compared to the deteriorations for the period 1992-1994. On the other hand, the IR index has increased substantially in the 2008-2009 period, while the MD ratio is very low in both periods. According to the values of the summary mobility indices, we may conclude that mobility has substantially increased in the post 2008 era, with the majority of transitions involving movements towards deterioration of capitalization.

The same conclusions are drawn, after observing the eigenvalue-based mobility indices. Recall that all eigenvalue-based mobility indices \( M_{PS} \), \( M_{SC} \) and \( M_S \) take on values in \([0, 1]\) interval; yet, some of them may take either very large values (\( M_S \)) or very small values (\( M_{SC} \)) and therefore our conclusions should be drawn by comparing their absolute values for the two periods rather than using as a mobility evidence their closeness to 1. The Prais-Shorrocks index (\( M_{PS} \)), reveals a medium level of mobility (55.31%) for the 1992-1994 transition matrix and much higher mobility (76.4%) for the 2008-2009 period. The \( M_{SC} \) index paints the same picture as the \( M_{PS} \) mobility index, if we take into account the fact that its value for the period 2008-2009 is much higher than the one exhibited for the period 1992-1994 (note that the respective ratio is about 0.0270/0.0008=33.75). The estimated half-life measure \( M_S \) has decreased substantially from 819.28 for the period 1992-1994 to 25.31 for the period 2008-2009. All aforementioned findings reveal that for the post 2008 era there is considerably higher mobility than in the period 1992-1994.

<table>
<thead>
<tr>
<th>Table 4.8. Mobility indices during the crises periods</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Index</strong></td>
</tr>
<tr>
<td><strong>Panel I: Summary Mobility Indices</strong></td>
</tr>
<tr>
<td>Immobility Ratio</td>
</tr>
<tr>
<td>Moving Up (deterioration)</td>
</tr>
<tr>
<td>Moving Down (improvement)</td>
</tr>
<tr>
<td><strong>Panel II: Eigenvalue Based Mobility Indices</strong></td>
</tr>
<tr>
<td>Prais-Shorrocks( (M_{PS}) )</td>
</tr>
<tr>
<td>Sommers-Conlisk( (M_{SC}) )</td>
</tr>
<tr>
<td>Shorrocks( (M_S) )</td>
</tr>
<tr>
<td>Half Life ( (h) )</td>
</tr>
</tbody>
</table>
4.6. CONCLUSIONS

Adopting the FDIC classification for US commercial banks' capital ratio, we estimate the transition probabilities of moving between capitalization buckets, including the default state. This is done for two sub periods - the 90's and the 00's. The estimated transition matrices were then used to compare the two decades in terms of persistence and mobility based on appropriate indices. According to our results we find that for both sub-periods the Adequately Capitalized and Well Capitalized banks exhibit the lowest transition probabilities towards default, in comparison to banks in any other capitalization bucket. However, there are substantial differences between the two sub-periods. In particular, for the period 1992-2000 the Adequately Capitalized and Well Capitalized categories transited to default with probability 0.66% and 0.02% respectively. In contrast, for the period 2001-2009 the corresponding probabilities showed a step increase, with Adequately Capitalized and Well Capitalized categories transiting to default with probabilities of 4.9% and 0.11% respectively. A similar increase of default probability transitions is observed for banks belonging to any of the undercapitalized categories. For instance, during the 1992-2000 period these probabilities were 5.49%, 29.93% and 20.45% for the Undercapitalized, Significantly Undercapitalized and Critically Undercapitalized buckets respectively, while for the period 2001-2009 the corresponding probabilities were 36.56%, 66.67% and 53.85%.

Looking at the mobility, defined as the tendency to change capitalization bucket, of the system based on a battery of mobility indices, we document a substantial increase in the post 2000 era. What is more important is that this mobility takes the form of increased probability of capitalization deterioration. Moreover, during the post 2001 period these deteriorations were more likely to transcend one capitalization bucket. That is during the 00's, in comparison to the 90's, the US banking sector has exhibited greater mobility, with capitalization deteriorations (including defaults) been more likely and more abrupt.

The analysis has several policy relevant findings. Firstly, it offers (annual) capitalization bucket-specific transition probabilities towards all possible states. This is useful for the regulator who can anticipate the likely movements across the capitalization spectrum, with special reference to the default state. Secondly, it
provides a comparison of capitalization movement experiences between the past two decades, therefore highlighting possible developing trends.

Future research could explore possible higher order time dependence of the Markov Chain. Additionally, one could allow for geographic and size dependent transition probabilities.
CHAPTER 5. THE LIFE CYCLE OF BANK FAILURE SIGNALS: EX ANTE FORECASTING WITH CAMEL RISK FACTORS

The purpose of this chapter is to investigate which bank metrics exhibit ex ante long term forecasting power for bank failures. In particular, using ex ante information on several key CAMEL\(^1\) factors we explore how many quarters prior to failure, they may contain useful signals for the excess vulnerability of banks. These signals do not pertain to specific banks, but rather to groups of banks categorized according to their exposure to certain risk factors that are assumed to drive failure. In addition, we quantify the excess vulnerability of specific groups across different time horizons.

5.1. INTRODUCTION

Bank supervisory agencies are responsible for monitoring the financial conditions of commercial banks and enforcing the relevant legislation and regulatory policy. A useful tool for accomplishing that task is the family of the so called CAMEL risk factors. The acronym “CAMEL” refers to the five components of a bank’s conditions that are assessed: Capital adequacy, Asset quality, Management ability, Earnings, and Liquidity.

The official CAMEL ratings are commonly viewed as typical summary measures of banks’ overall financial condition. Ratings are assigned for each component on a scale from 1 to 5 in addition to the overall rating of a bank’s financial condition. Banks with ratings of 1 or 2, initiate few, if any, supervisory concerns, while banks with ratings of 3, 4, or 5 indicate moderate to extreme degrees of supervisory concern.

The general consensus in financial literature is that the information contained in CAMEL ratings can be efficiently exploited for the supervisory monitoring of banks. Since this information is usually disclosed to the financial markets (and therefore affects the prices of bank

\(^1\) CAMEL is the acronym for Capital Adequacy, Asset Quality, Management Ability, Earnings, and Liquidity.
securities), the supervisory information in CAMEL ratings appears to be useful in the public monitoring of banks as well.

An issue of major importance in bank supervision is, which risk factors contain useful information for bank failure. With respect to predicting bank default, Barker and Holdsworth (1993) reported that CAMEL ratings are providing substantial information, about the condition and performance of banks. On the other hand, Cole and Gunther (1998) dealt with a similar question and found that, the information in supervisory CAMEL assessments deteriorates quickly over time. More specifically, analyzing data for the period 1988-1992, they found that new (less than 6 months old) CAMEL ratings more accurately predict bank financial distress than financial ratios can; however, financial ratios are better predictors than older (more than 6 months old) CAMEL ratings. O’Keefe and Dahl (1996) concluded that this result is asymmetric in the sense that CAMEL ratings become less reliable over time for banks with deteriorating finances, but not for banks with improving financial condition.

DeYoung, et al., (1998) examined whether private supervisory information is useful in pricing the subordinated debt of large bank holding companies (BHCs). They use an econometric method to estimate the private information component of the CAMEL ratings for the BHCs’ lead banks and regressed it onto subordinated bond prices. They found that this approach adds significant explanatory power to the regression after controlling for publicly available financial information and that it is incorporated into bond prices about six months after an exam.

Berger and Davies (1998) used CAMEL ratings to examine the behavior of BHC stock prices in the eight-week period following an exam of its lead bank and found out that CAMEL downgrades reveal unfavorable private information about bank conditions to the stock market. Berger, Davies, and Flannery (1998) extended this analysis to investigate whether the information about BHC conditions gathered by supervisors is different from that used by the financial markets. More specifically they applied Granger causality analysis to the leading and lagged relationships between exam ratings and the actions of bank stakeholders in financial markets for 184 bank holding companies between 1989 and 1992. Their main finding was that lagged movements in BOPEC ratings (the safety and soundness ratings for bank holding companies) explain 1.6 percent of the ‘additional’ variation in shareholder market variables (i.e.,
stock returns, changes in insider and institutional shareholdings), but explain 4.1 percent of the ‘additional’ variation in bond ratings.

Hirtle and Lopez (1999) scrutinized on the usefulness of past CAMEL ratings in assessing banks’ current conditions. Their main finding was that, the private supervisory information contained in past CAMEL ratings provides further insight into bank’s current conditions, as summarized by current CAMEL ratings. The authors used data for the period from 1989 to 1995 and concluded that, the supervisory information gathered during the last on-site exam remains useful with respect to the current condition of a bank for up to 1.5 to 3 years.

Dincera et al. (2012) used the CAMEL ratios to illuminate the effects of the global 2008 crisis on the performance of Turkish banking sector. In addition they exploited them to study the trends before and after the crisis in the Turkish economy. Kabir and Dey (2012) studied the comparative performance of two leading private sector commercial banks in Bangladesh, on the basis of the CAMEL rating system.

In summary, we may state that the overall conclusion drawn from the related academic literature is that the supervisory information, as summarized by CAMEL ratings, is quite helpful for the supervisory monitoring of bank conditions.

In the present study we employ a panel dataset whose cross section consists of all US commercial banks insured by the FDIC and its time series dimension of the period 2000:q1-2012:q1 (quarter intervals). Our aim is to investigate which CAMEL risk factors exhibit long term forecasting ability for bank failures. In particular, we explore how many quarters prior to failure they may contain useful signals for the excess vulnerability of banks. The novelty of the empirical analysis is that it focuses on signals pertaining not to specific banks, but rather to groups of banks exposed to certain level of the risk factors. Our empirical findings are particularly useful both for market participants, but especially for the regulator.

Let us briefly discuss our main empirical findings. As far as the CAMEL metrics are concerned, it appears that there are no substantial differences between (subsequently) failed banks and their non-failed peers up until about 20 quarters prior to failure. However, starting approximately 20 quarters before failure, the relevant metrics of (subsequently) failed banks progressively deteriorate leading to an apparent divergence between the two groups. Four to five
years prior to failure one can easily verify that 3 (out of the 11) risk factors contain significant forecasting power. Looking at a 3 year horizon before failure we uncover that the set of significant predictors becomes larger. Finally, the empirical results across CAMEL metrics two years prior to failure indicate (as expected, given the proximity to failure) that most of the risk factors (10 out of 11) contain significant forecasting power. Based on the above findings, we suggest breaking up the life cycle prior to failure to following three stages: 3-5 years prior to failure, 3 years prior to failure and finally 1 year prior to failure.

The remainder of this chapter is organized as follows. Section 5.2 describes the data used for the analysis while Section 5.3 presents the empirical methodology followed. In Section 5.3.1 we present the procedure of defining four risk exposure classes for each CAMEL factor and conduct a preliminary exploratory analysis (by the aid of simple graphs and tables) based on the distribution of future failures in the 4 risk exposure classes. In Section 5.3.2 we use the Correspondence Analysis technique to investigate the connection between bank failure and the categorical variables describing the four exposure levels of the CAMEL risk factors under a multivariate framework. In Section 5.3.3 we describe how a logistic regression model can be built and used for studying bank failures over specific time horizons. Focusing on 5 time horizons, we built in Section 5.3.4, different logit models for each risk factor and horizon, containing as explanatory variables the three higher risk exposure classes (having the lowest risk exposure class as a reference group). Combining the findings of the Correspondence Analysis approach and the logistic regression techniques, we substantiate that, according to the group of CAMEL factors that send out signals for bank failure, we may breaking up the bank life cycle (prior to failure) to 3 stages. The suggested grouping of CAMEL factors is further analyzed by exploiting Discriminant Analysis techniques in Section 5.3.5, and the results are reassuring the existence of the 3 stages. In Section 5.3.6 we estimate, using the logistic models developed in Section 5.3.4, the probabilities of bank failure per exposure class for the significant CAMEL risk factor of each stage. In Section 5.3.7 we examine the problem of developing a single index for failure forecasting in each of the three stages of bank failure lifecycle. This is achieved by applying PCA and creating logit models using as explanatory variables dummy variables associated to the quartiles of the linearly transformed data. Finally, in Section 5.4 we give an overview of the main findings of the present chapter.
5.2. DATA ISSUES AND BACKGROUND ANALYSIS

The sample consists of all FDIC-insured commercial banks in USA, for which quarterly data covering the period 2000:q1-2011:q4 are obtained. The lion share of bank defaults during this period has taken place in the post-2008 era, essentially as the result of the financial crisis. The period 2000-2007 presents a relatively calm time span with sporadic bank failures\(^2\). These failures are excluded from the sample because they do not lend themselves to long term forecasting. Figure 5.1 and 5.2 depict the number of defaults per quarter and the total number of defaults per year in the post-2008 period.

![Bar chart showing number of bank failures per quarter](image)

**Figure 5.1.** Number of bank failures per quarter

In this chapter we are examining whether failure predictions can be inferred from the following widely used in the literature, CAMEL factors, which are described in Table 5.1. In order to check the decay of the risk factors when we move away from the default time, let us first proceed to a preliminary analysis by the aid of simple graphs. Figures 5.3-5.5 depict the values of the first 3 CAMEL risk factors (rbc, ncl and equity) for a period up to 42 quarters before the reference point (i.e. failure, for the banks that defaulted).

Figure 5.2. Total number of bank failures per year

Table 5.1. CAMEL factors’ abbreviation and computational formula.

<table>
<thead>
<tr>
<th>CAMEL Factor</th>
<th>Abbreviation</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk Based Capital</td>
<td>rbc</td>
<td>(Total risk based capital / risk weighted assets)*100</td>
</tr>
<tr>
<td>Non-current loans</td>
<td>ncl</td>
<td>(Non-current loans / total loans)*100</td>
</tr>
<tr>
<td>Equity</td>
<td>equ</td>
<td>(Equity / total assets)*100</td>
</tr>
<tr>
<td>Return on assets</td>
<td>roa</td>
<td>(Net income after taxes and extraordinary items / total assets)*100</td>
</tr>
<tr>
<td>Coverage</td>
<td>cov</td>
<td>(Income before income taxes and extraordinary items and other adjustments + provisions for loan and lease losses and allocated transfer risk reserve + plus gains on securities not held in trading accounts)/ net loan and lease charge-offs</td>
</tr>
<tr>
<td>Net interest margin</td>
<td>nim</td>
<td>(Net interest income /total assets)*100</td>
</tr>
<tr>
<td>Efficiency</td>
<td>eff</td>
<td>(Non-interest expense - the amortization expense of intangible assets / net interest income + noninterest income)*100</td>
</tr>
<tr>
<td>Net charge offs</td>
<td>nco</td>
<td>(Net charge offs / total loans)*100</td>
</tr>
<tr>
<td>Risk weighted assets</td>
<td>rwa</td>
<td>(Risk weighted assets / total assets)*100</td>
</tr>
<tr>
<td>Loans 30 days past due date</td>
<td>past30</td>
<td>(Loans up to 30 days past due / total loans)*100</td>
</tr>
<tr>
<td>Loan growth</td>
<td>lgr</td>
<td>Annual growth rate of total loans and leases.</td>
</tr>
</tbody>
</table>

The graphs clearly indicate that, as far as the requested metrics are concerned, there are no substantial differences between (subsequently) failed banks and their non-failed peers up until about 20 quarters prior to failure. However, starting approximately 20 quarters before failure the relevant metrics of (subsequently) failed banks progressively deteriorate leading to an apparent
divergence between the two groups. The picture conveyed by the rest CAMEL factors is quite similar, see Figures 5.6-5.9.

Thus, a 20-quarter horizon seems a good starting point for investigating the long term forecasting ability of the CAMEL metrics over bank failure. This is why, in the subsequent analysis, only time periods shorter than a 20-quarter horizon will be analyzed. More specifically, we are going to consider time horizons of 5 years (20 quarters), 4 years (16 quarters), 3 years (12 quarters), 2 years (8 quarters) and 1 year (4 quarters).

**Figure 5.3.** Graph of rbc for non-failed (smooth line) and failed (dotted line) banks up to 42 quarters before failure

![Graph of rbc for non-failed (smooth line) and failed (dotted line) banks up to 42 quarters before failure](image)

Notes: On the horizontal axis -40, -30, -20 and -10 denote quarters prior to failure.

**Figure 5.4.** Graph of ncl for non-failed (smooth line) and failed (dotted line) banks up to 42 quarters before failure

![Graph of ncl for non-failed (smooth line) and failed (dotted line) banks up to 42 quarters before failure](image)

Notes: On the horizontal axis -40, -30, -20 and -10 denote quarters prior to failure.
Figure 5.5. Graph of equ for non-failed (smooth line) and failed (dotted line) banks up to 42 quarters before failure.

Notes: On the horizontal axis -40, -30, -20 and -10 denote quarters prior to failure.

[Figures 5.6-5.9 in Appendix]

5.3 EMPIRICAL METHODOLOGY

Let us briefly describe our empirical tactic. The thrust of our analysis will be the use of appropriate logistic regression models containing as explanatory variables the three higher risk exposure classes of each CAMEL risk factor (having the lowest risk exposure class as a reference group). These models will assist us in identifying which risk exposure class is more (less) vulnerable to failure for each time horizon, as reflected in failure rates between risk exposure classes. The assessment of the comparison of each exposure class to risk class 1 (baseline) will be carried out by the odds ratios after the estimation procedure is completed for each time horizon.

Three popular multivariate techniques, namely Correspondence Analysis, PCA and Discriminant Analysis will also be used to facilitate our initial target, which pertains to the investigation of bank life cycle (prior to failure).

Before applying the aforementioned statistical techniques, a preliminary analysis of the available data will be carried out in order to find out at which time horizon the CAMEL metrics we are focusing on start carrying useful information on future failures (defaults). This step is
deemed necessary since, as reported in the relevant past literature in this area, the evidence provided by the CAMEL risk factors for a future default deteriorates quickly over time.

5.3.1. Preliminary statistical analysis

Let $F_j$ denote the $j$th risk factor ($j = 1, 2, ..., 11$) from the family of CAMEL factors mentioned in the previous section. For each risk factor $F_j$ we use the observed quartiles to construct four exposure levels, say $E'_j$ where $j$ indicates the risk factor and $r$ ($r = 1, 2, 3, 4$) denotes the four classes generated by the quartiles, in increasing risk exposure. From now on we shall refer to the $E'_j$ by the term $r$-th (risk) exposure class of the $j$th risk factor.

Apparently, when different time horizons are considered (length of the period before the reference point i.e. failure, for the banks that defaulted), the quartiles will change.

Using the panel data described in the previous Section, we calculated the cut-points (quartiles) for the risk exposure classes of each risk factor and time horizons 5, 4, 3, 2 and 1 years before the reference point. The results are summarized in Table 5.2. Apparently the changes in the cut-points when shifting away (backwards) from the reference point are not very large.

Table 5.3 can be used for a rough comparison of different factors in predicting bank failures for the 5 time horizons we are interested in. By the final quarter of the time period examined (2000 - 2011), each bank either never failed or had failed at some time point, not necessarily at the last quarter of 2011. Looking backwards for a specific period before the reference point (e.g. failure time, for the banks that defaulted), we may classify the banks to one of the exposure classes mentioned before, according to the cut-points shown in Table 5.2.

Panel (a) of Table 5.3 compares the 11 CAMEL risk factors as predictors of bank failures for a time period of five years forward. The 11 factors analyzed are listed across the top of the panel. For each factor $j = 1, 2, ..., 11$, the banks are sorted from lowest to highest and divided into 4 equal portions (the four exposure levels $E'_r$, $r = 1, 2, 3, 4$), with banks in exposure level 1 supposedly having the lowest risk and those in exposure level 5 having the highest. By reading across each exposure level, one can see how many banks failed for each risk factor. For instance, out of the 339 banks that have failed in the period
examined, 42 belonged to the first exposure level for the risk based capital ratio (rbc) as formed 5 years before their failure, 37 belonged to the second exposure level, 81 belonged to the third exposure level and finally 179 belonged to the fourth exposure level.

Table 5.2. Cut-points (quartiles) for the exposure classes of the Camel factors for all time horizons

<table>
<thead>
<tr>
<th>Factor quartile</th>
<th>Factor</th>
<th>rbc</th>
<th>ncl</th>
<th>equ</th>
<th>roa</th>
<th>cov</th>
<th>nim</th>
<th>eff</th>
<th>past30</th>
<th>nco</th>
<th>rwa</th>
<th>lgr</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>rbc</td>
<td>ncl</td>
<td>equ</td>
<td>roa</td>
<td>cov</td>
<td>nim</td>
<td>eff</td>
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On the other hand, out of 346 failed banks that have been analyzed with respect to the non-current loans risk factor (nco) - the slight difference in the total number analyzed for each factor is due to missing data- 125 belonged to the first exposure level for nco as formed 5 years before their failure, 75 belonged to the second exposure level, 74 to the third exposure level and finally 72 belonged to the fourth exposure level. From these figures it is intuitively clear that the explanatory power of rbc, as predictor of failure in a 5 years’ time horizon, is much higher that nco. Below the Total row, we have provided the Chi-Square statistic as calculated when analyzing the contingency table arising from the failed/non-failed banks distribution across the 4 exposure classes (for each risk factor). The higher the Chi-Square statistic, the better the risk factor is a predictor of bank failures.

Apparently, out of the 11 risk factors studied there are three whose Chi-square statistic is significantly higher that the others’: rwa, lgr and rbc. Rwa is the risk factor with the highest Chi-square (341.1), lgr is the one with the next-highest score (174) and rbc has the third highest Chi-square statistic value (153.7). Therefore, those three factors seem to be by far the best predictors for a failure in a 5 years’ time horizon.

The numerical results commended so far are also displayed graphically in Figure 5.10a where the number of defaults per exposure class of each Camel factor are shown for a 5 years’ time horizon. From this figure it is clear that the highest contrast between exposure classes 3-4 (as compared to classes 1-2) is exhibited in the three risk factors mentioned earlier (rwa, lgr and rbc).

For a time period of 4 years ahead, the picture conveyed from Panel (b) of Table 5.3 and Figure 5.10b is exactly the same as for the 5 years period.

For a forecasting period of 3 years, as seen in panel (c) of Table 5.3, rwa, lgr and rbc retain their predicting power. In addition, 2 more factors seem to kick in, namely roa and ncl, with Chi-square statistics 92 and 57.9 respectively. One might think of eff as a nice predictor, as conferred from the high Chi-square value associated with it (44.2). However, looking at the distribution of the defaults along the risk classes, it is apparent that this is not quite true, since a large number of defaulted banks belong to the first risk exposure class as well; Thus, the large Chi-square value (44.2) reflects the contrast between classes 1 and 4 versus 2 and 3. At this point, one may comment that the risk factor cov, despite the lower Chi-square value (36.1), as compared to eff, may serve as a better predictor for a 3 year horizon failure predictor, since the risk class 4 of this
factor is by far larger than any other risk class of the same factor. Figure 5.10c clearly illustrates all the aforementioned findings for a 3-year period.

### Table 5.3. Comparison of failures per exposure class of Camel factors for all horizons

#### a. 5 years forecasting horizon

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<th>roa</th>
<th>cov</th>
<th>nim</th>
<th>eff</th>
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</table>

100
Panels (d) and (e) of Table 5.3 summarize the results for the 2-year and 1-year horizon respectively. For these periods all CAMEL factors gain a substantial predicting power, with the notable exception of lgr who apparently loses its forecasting ability over near future failures. Figures 5.10c and 5.130d clearly illustrate these statements. Figures 5.11a-5.11h display the distribution of failures of defaults per exposure class for each of the 11 Camel factor, across the 5 horizons. For a bigger illustration of Figures 5.10. and 5.11 see Appendix .

Based on the above findings we may conclude that indeed there are CAMEL factors that contain significant ex ante explanatory power over future bank failures. This suggests that particular groups of banks can be identified as being ex ante more susceptible to failure with distinctly higher probability of failure. Another important finding is that one may probably distinguish different stages based on the groups of CAMEL risk factors from which failure signals are received. We shall come back to this issue in the next Sections of this Chapter .
Figure 5.10. Comparison of defaults per exposure class of Camel factors for each horizon

(a) 5 years horizon
(b) 4 years horizon
(c) 3 years horizon
(d) 2 years horizon
(e) 1 year horizon
Figure 5.11. Comparison of defaults per exposure class for each Camel factor, across the 5 horizons

CAMEL factor: rbc

CAMEL factor: ncl

CAMEL factor: equ

CAMEL factor: roa

CAMEL factor: cov

CAMEL factor: nim
Figure 5.11. (Continued) Comparison of defaults per exposure class for each Camel factor, across the 5 horizons
5.3.2 Correspondence Analysis for Risk Exposure Classes

Multivariate data arise when one records the values of several characteristics (random variables) on a number of subjects or units in which she is interested in. This leads to a vector-valued or multidimensional observation for each unit. Such cases arise in a wide spectrum of disciplines, and it is probably reasonable to claim that the majority of data sets met in practice are multivariate. In a financial framework, when studying a bank, the measurements we gather on each one consist of a bunch of different characteristics. The set of CAMEL factors studied in the previous paragraphs offers a typical example of multivariate data, with each factor representing a characteristic (random variable), and the whole set of factors forming the vector-valued or multidimensional observation for each bank. Therefore, the nature of the dataset described in Section 5.2 is in fact multivariate, and its investigation will apparently call for the use of multivariate analysis methods.

One of the problems with multivariate data is that they are usually comprised of too many variables to make the application of the graphical techniques (as those described in Section 5.3.1) successful in providing an informative assessment of the majority of the data, especially when their interrelation is of special interest. The possible problem of simultaneously analyzing too many variables is sometimes known as the curse of dimensionality, see Bellman (1961).

This problem has motivated the search for multivariate techniques aiming at the reduction of the dimensionality of a multivariate data set while retaining as much of the information present in original the data set as possible. In statistical terminology the information contained in a data set is simply the overall variation of the characteristics being studied. Two of the most popular multivariate techniques that have been developed in this direction are Principal Components Analysis and Correspondence Analysis.

This aim of Principal Components Analysis is to transform the initial variables to a new set of variables, the principal components, which are linear combinations of the original variables. The new set consists of variables that are orthogonal and are ordered so that the first few of them account for most of the variation in all the original variables.

Correspondence analysis is analogous to principal components but operates on categorical data arranged in the form of the contingency tables. Usually it is exploited for two-way
contingency tables to obtain a graphical representation of the residuals from the independence model and identify possible interrelations between the categories of the variables under investigation. For more than two categorical variables one can use the multiple correspondence analysis technique (for a book length presentation see Greenacre (1983)). We shall give shortly some details on the correspondence analysis framework and its tools and use it for analyzing the CAMEL factor categories (quartiles) defined in Section 5.3.1. An analysis of our data exploiting the Principal Components Analysis machinery will be presented later on, in Section 5.3.6.

The genesis of multivariate analysis may be attributed to the work carried out by Francis Galton and Karl Pearson in the late 19th century in an effort to quantify the relationship between offspring and parental characteristics (multiple regression theory) and the development of the correlation coefficient. Later on, in the early years of the 20th century, Charles Spearman, motivated by the effort to investigate correlated intelligence quotient tests, laid down the foundations of factor analysis. Spearman's work was subsequently extended by Hotelling, Fisher and other famous statisticians.

Multivariate methods were also inspired by problems in various scientific areas. In the 1930s Fisher developed linear discriminant function analysis to solve a taxonomic problem using multiple botanical measurements. At these early days, the developments were primarily mathematical, due to the fact that the computational power needed for the vast amounts of calculations involved in the multivariate methods was very limited; as a consequence the multivariate research was, at the time, largely a branch of linear algebra.

In the second half of the 20th century, the arrival and rapid expansion of the use of electronic computers, especially PC’s, made the practical application of existing methods of multivariate analysis very popular and renewed interest in the creation of new techniques. In the early years of the 21st century, the wide availability of cheap powerful personal computers, and the development of statistical software offered the researchers the capacity to apply all the methods of multivariate analysis even to very large data sets arising in genetics, finance, astronomy etc. The application of multivariate techniques to large data sets has now been given its own name, data mining, and is a rapidly expanding research area.

In this section we are going to use the Correspondence Analysis technique to assess the relationship between bank failure and the categorical variables describing the four exposure
levels $E_j^r$, $r=1,2,3,4$ of the CAMEL risk factors. This will be done in a multivariate framework, that is we shall be analyzing all risk factors simultaneously, a task that cannot be achieved by simple two-way contingency tables and the related statistics.

Before proceeding to the analysis of our data, let us present some introductory material for the Correspondence Analysis technique. Correspondence analysis provides tools for analyzing the associations between rows and columns of contingency tables. A contingency table is a two-entry frequency table where the joint frequencies of two qualitative variables are reported. Correspondence Analysis has the potential to display graphically the relationships for multiple two-way tables of counts, by deriving coordinates representing the row categories and column categories of the table. The coordinates for each category are analogous to those derived from principal components of continuous data, except that they are couched in a decomposition of the total $\chi^2$ value used in testing independence, rather than the total variance (which is the quantity decomposed in used in Principal Components Analysis). For the mathematical models underlying in the Correspondence Analysis theory, the interested reader is referred to Benzécri, J.-P. (1973) and the more recent monographs of Greenacre (1983), (2007).

Simple correspondence analysis is performed on a $m \times n$ contingency table where $m$ is the number of distinct values of a categorical variable and $n$ is the number of distinct values of a second categorical variable. The aim of correspondence analysis is to develop simple indices that show relations between the row and columns of a contingency table. The method produces orthogonal components for each item in the contingency table and a set of scores (sometimes called factor scores) and useful indices that can be exploited for the interpretation of the results.

The graphical relationships between the rows and the columns of the table that results from correspondence analysis are based on the idea of representing all the row and column categories and interpreting the relative positions of the points in terms of the weights corresponding to the column and the row. This is achieved by deriving a system of simple indices providing the coordinates of each row and each column. These row and column coordinates are simultaneously represented in the same graph.

Correspondence Analysis is a data analysis algorithm that is similar to Principal Components Analysis. They both construct a low-dimensional Euclidean output, that can be
visualized as a type of map. The mathematics behind both is also very similar: the factors in correspondence analysis, and the principal components in principal components analysis, are defined from the eigenvectors of a square, symmetric matrix. While Principal Components Analysis is particularly suitable for quantitative data, correspondence analysis is appropriate for the following types of input data: frequencies, contingency tables, probabilities, categorical data, and mixed qualitative/categorical data.

Correspondence Analysis identifies appropriate axes to which the cloud of points are as close as possible. This criterion is equivalent to the following: that the projections of points on any axis should be as dispersed as possible, i.e. that the variance of the projections be large. When mass or weight is considered, variance takes the form of inertia, and it is this latter term that is used in correspondence analysis.

When studying the relations between many categorical variables, it is rare to find that they approximately lie on a one-dimensional surface, i.e., a line. Then the need arises for a second best-fitting axis, orthogonal to the first already found; these axes will together constitute a best-fitting plane (2-dimensional factorial space), where the units can be displayed and observed.

While we have described the method of Correspondence Analysis as a tool for analyzing two categorical variables, one can extend the approach to cover the general case of examining many variables. The principal steps in interpreting the output of multiple correspondence analysis are as for the case of correspondence analysis.

It frequently happens that there are additional characteristics that are not the primary data to be analyzed by the Corresponding Analysis technique (i.e. the ones producing the factorial axes) but that are useful in interpreting features discovered in the primary data. Any additional such characteristic can be positioned on an existing map, so as a meaningful comparison to the primary characteristics is carried out. These additional categorical variables are called supplementary variables while the points projected on the factorial plane, are usually termed as points.

Interpretation of the outcomes of Correspondence Analysis is usually limited to the first few factors. A quantity termed as “quality” accounts for the overall quality of representation of
the data categories on the factorial space. The typical steps in interpreting the results of Correspondence Analysis are usually the following:

1. Projections on the most significant factorial axes (usually 1, 2 and 3) and on the respective planes formed by them (i.e. plane created by the first and second factorial axes, plane created by the second and third factorial axes and probably the one formed by the first and third one) of the variables that are analyzed along with their values. Moreover, one may proceed to the projection of any other supplementary variables points that he/she wishes to associate with them.

2. Spectrum of non-increasing values of eigenvalues, which provide information about the overall quality of representation on the factorial axis. The quality of representation on the factorial plane is simply the sum of the qualities of each axis.

3. Interpretation of axes. Usually contrast is important: what is found to be analogous shares the same sign in the axis, while non-analogous elements are placed at areas with different signs (polarities).

5. The values of squared correlation (sqcorr) are squared cosines, which can be considered as being like correlation coefficients from the factors towards the elements. If sqcorr is high we can say that that element is well explained by the respective factorial axis.

For detailed discussions of the interpretation of Correspondence Analysis plots (usually referred as biplots) the interested reader may consult Benzécri (1992) and Greenacre (2007). Briefly, we can say the following. The orientation of the axes in the biplot are determined by the coordinates of eigenvectors, so it may be considered arbitrary (we recall that if we multiply all the coordinates by the same non-zero number, we get an eigenvector as well). If the point representing one row or column (categories of the qualitative characteristics we study) falls far from the origin, it will generally be opposed by one or more points in the opposite direction. The actual linear combinations of the values contributing to each point are not very important, but the relative positions and the shape of the overall plot may be very informative for the structure and the occurrence of dependencies between the characteristics and their influence to supplementary categorical variables.
Tables 5.4 and Table 5.5 (see Appendix) summarize the results derived by the Correspondence Analysis technique on the group of 11 CAMEL risk factors as recoded with the use of their quartiles at a time horizon of 20 quarters (5 years) before failure. Table 5.4 indicates the proportion of variance (inertia, recall that in Correspondence Analysis the variance of categorical data is measured by a chi-square statistic) which is explained by the first 10 factorial axes (dimensions). According to Table 5.4, the first factorial axis accounts for 26.12% of the total variance, the second for 19.40% and the third for 11.27%. The percentages of the rest axes are decreasing very quickly and in fact become insignificant. Using the plane determined by the first two factorial axes, we can get the 45% of the variance in our data (see column labeled as “cumulative percent”), which will be sufficient for extracting some preliminary results on the relation of the CAMEL factors risk class to the failure of a bank in a 5 time horizon. In some cases, when no clear conclusion can be drawn from the first two axes the third axis may facilitate our study.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Principal Inertia</th>
<th>Percent</th>
<th>Cumulative Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0211709</td>
<td>26.12</td>
<td>26.12</td>
</tr>
<tr>
<td>2</td>
<td>0.0157218</td>
<td>19.40</td>
<td>45.52</td>
</tr>
<tr>
<td>3</td>
<td>0.0091348</td>
<td>11.27</td>
<td>56.79</td>
</tr>
<tr>
<td>4</td>
<td>0.004941</td>
<td>6.10</td>
<td>62.89</td>
</tr>
<tr>
<td>5</td>
<td>0.0027248</td>
<td>3.36</td>
<td>66.25</td>
</tr>
<tr>
<td>6</td>
<td>0.0013594</td>
<td>1.68</td>
<td>67.93</td>
</tr>
<tr>
<td>7</td>
<td>0.0003185</td>
<td>0.39</td>
<td>68.32</td>
</tr>
<tr>
<td>8</td>
<td>0.0002392</td>
<td>0.30</td>
<td>68.62</td>
</tr>
<tr>
<td>9</td>
<td>0.0001291</td>
<td>0.16</td>
<td>68.78</td>
</tr>
<tr>
<td>10</td>
<td>0.0000542</td>
<td>0.07</td>
<td>68.84</td>
</tr>
<tr>
<td>Total</td>
<td>0.0810433</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.5 in Appendix displays the coordinates of each risk category of all 11 CAMEL factors, along with the typical quality metrics which are traditionally used to assess the quality of representation of a point at the factorial axes and the (3-dimensional) space formed by them. Two metrics are of special interest: “sqcorr” and “overall quality”. The values of sqcorr are squared cosines, which can be considered as being correlation coefficients from the factors towards the elements (they are in fact squared correlation). If sqcorr is high we can derive that element is well explained by the respective factorial axis. The “overall quality” is the sum of squares of the three sqcorr’s and gives the squared correlation (squared cosine) between the original variables and the factorial space created by the first, second and third factor axis. If the
overall quality is high we can state that the respective CAMEL factor exposure class (we recall that the factors have now been recoded in four exposure classes according to the 3 quartiles), is well explained by the respective factorial space.

Table 5.5. Correspondence analysis coordinates for exposure class 4 and quality metrics for a 5 year horizon (shortened version of Table 5.5 given in Appendix)

<table>
<thead>
<tr>
<th>Categories</th>
<th>overall quality</th>
<th>dimension 1</th>
<th></th>
<th></th>
<th>dimension 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>coord</td>
<td>sqcorr</td>
<td>Coord</td>
<td>sqcorr</td>
<td>Coord</td>
<td>sqcorr</td>
</tr>
<tr>
<td>rbc</td>
<td>0.545</td>
<td>-0.882</td>
<td>0.114</td>
<td>-1.812</td>
<td>0.357</td>
<td></td>
</tr>
<tr>
<td>ncl</td>
<td>0.821</td>
<td>1.567</td>
<td>0.732</td>
<td>-0.201</td>
<td>0.009</td>
<td></td>
</tr>
<tr>
<td>equ</td>
<td>0.559</td>
<td>-0.361</td>
<td>0.034</td>
<td>-1.203</td>
<td>0.281</td>
<td></td>
</tr>
<tr>
<td>roa</td>
<td>0.669</td>
<td>2.279</td>
<td>0.473</td>
<td>-0.036</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>cov</td>
<td>0.675</td>
<td>2.519</td>
<td>0.610</td>
<td>-0.878</td>
<td>0.055</td>
<td></td>
</tr>
<tr>
<td>nim</td>
<td>0.696</td>
<td>0.840</td>
<td>0.167</td>
<td>1.192</td>
<td>0.249</td>
<td></td>
</tr>
<tr>
<td>eff</td>
<td>0.672</td>
<td>2.083</td>
<td>0.470</td>
<td>-0.038</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>past30</td>
<td>0.721</td>
<td>0.980</td>
<td>0.458</td>
<td>-0.360</td>
<td>0.046</td>
<td></td>
</tr>
<tr>
<td>nco</td>
<td>0.592</td>
<td>1.660</td>
<td>0.466</td>
<td>-0.880</td>
<td>0.097</td>
<td></td>
</tr>
<tr>
<td>rwa</td>
<td>0.515</td>
<td>-0.855</td>
<td>0.172</td>
<td>-1.373</td>
<td>0.329</td>
<td></td>
</tr>
<tr>
<td>lgr</td>
<td>0.749</td>
<td>-0.569</td>
<td>0.173</td>
<td>-0.757</td>
<td>0.228</td>
<td></td>
</tr>
<tr>
<td>def</td>
<td>-1.458</td>
<td>-1.517</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ndf</td>
<td>0.002</td>
<td>0.003</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In the following tables and graphs, the exposure classes of the 11 CAMEL factors have been prefixed with a number indicating the factor, according to the numbering shown in Table 5.6.

Table 5.6. CAMEL factor numbering used in Correspondence Analysis tables and graphs

<table>
<thead>
<tr>
<th>Number</th>
<th>CAMEL factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Rbc</td>
</tr>
<tr>
<td>2</td>
<td>Ncl</td>
</tr>
<tr>
<td>3</td>
<td>Equ</td>
</tr>
<tr>
<td>4</td>
<td>Roa</td>
</tr>
<tr>
<td>5</td>
<td>Cov</td>
</tr>
<tr>
<td>6</td>
<td>Nim</td>
</tr>
<tr>
<td>7</td>
<td>Eff</td>
</tr>
<tr>
<td>8</td>
<td>past30</td>
</tr>
<tr>
<td>9</td>
<td>Nco</td>
</tr>
<tr>
<td>10</td>
<td>Rwa</td>
</tr>
<tr>
<td>11</td>
<td>Lgr</td>
</tr>
</tbody>
</table>

So in the “categories” column of Table 3.3.2.2 in Appendix C the first exposure class of rbc has been labeled as “11”, the second as “12”, the third as “13” and the fourth as “14”. Likewise, the first exposure class of rwa has been labeled as “101”, the second as “102”, the
third as “103” and the fourth as “104”. Although this add-on in the exposure levels categories does not offer any particular help in reading the tables (since the name of the factor is explicitly stated in the table), it is very helpful when the graphs are inspected, since the number of points displayed therein is quite large and the colors used to plot each factor are not easily distinguished (see e.g. Figure 5.12a which will be commended later on).

As stated before, Table 5.5 given in Appendix displays the coordinates of each risk category of all 11 CAMEL factors in the first 3 factorial axes. At the end of the table we have the coordinates of the binary supplementary variable that describes whether a bank has defaulted or not 5 years after the specific CAMEL factor data were collected. The coordinates of the defaulted banks are -1.458 in the first factorial axis -1.517 in the second and -1.620 in the third, while the respective coordinates of the non-defaulted banks are 0.002 in the first factorial axis, 0.003 in the second and 0.003 in the third.

In order to simplify the analysis we present below a shortened version of Table 5.5 of Appendix where we have kept values of the two useful metrics mentioned earlier (“sqcorr” and “overall quality”), the coordinates of the exposure class 4 of each risk factor and the coordinates of the defaulted/non-defaulted banks. Since the plane formed by the first two factorial axes is providing the best information for the variables under investigation (in addition a visual illustration is feasible in 2 dimensions) we provide only the coordinates in the first two factorial axes only.

It is of great importance that both coordinates of the point associated to the defaulted banks are negative (-1.458 in the first factorial axis and -1.517 in the second). This fact indicates that a negative sign in both coordinates of an exposure class reveals a close relationship between this class and the bank failure.

Out of the 11 CAMEL factors there are four whose both factorial coordinates are negative for the exposure class 4: rbc with coordinates (-0.882, -1.812), equ with coordinates (-0.361, -1.203), rwa with coordinates (-0.855, -1.373) and lgr with coordinates (-0.569, -0.757). Therefore, these exposure classes (of the respective CAMEL factors) are highly associated to bank failure, in the sense that, a bank belonging to these classes has increased probability of defaulting in a 5 years horizon.
It is worth noticing that, for these factors all three coordinates for the exposure class 3 are negative too, while the coordinates of exposure classes 1 and 2 are positive, a fact reinforcing the previous statement (see Table 5.5 in Appendix). Looking at the remaining seven CAMEL factors one can easily verify that no other factor displays the aforementioned contrasts between the coordinates of the exposure classes 1, 2 and the exposure classes 3, 4 (cf. Table 5.5 in Appendix). Although the equity ratio factor (equ), seems to have a similar behavior with rbc, rwa and lgr, the quality of representation of the 4 exposure classes in the first factorial axis is very low. This means that we should avoid using the positioning (coordinates) of the 4 classes in the first factorial axis to arrive at conclusions pertaining to the proximity of them to the point representing the bank failure.

Figure 5.12 displays graphically the points associated with each category of every CAMEL factor on the plane created by the two first axes of the Correspondence Analysis technique. The points representing the binary supplementary variable that describes whether a bank has defaulted on not 5 years after the specific CAMEL factor data were collected, is placed on the same plane as well; the state “defaulted” is represented by the point which is labeled as “def”, whereas the “non-defaulted” state labelled as “ndf”. It is clear that the points associated with the exposure class 4 of CAMEL factors 1 (rbc), 10 (rwa) and 11 (lgr), which points are labeled as 14,104 and 114, are very close to the point “def”.

Continuing the point labeled as 34 is also close to the point indicating bank failure, however, this is related to the CAMEL risk factor 3 (equ) whose quality of representation, especially in the first axis is very low, so any conclusion based on its positioning may be distrusted.

In order to make the picture conveyed by Figure 5.12 more clear, we have created Figure 5.13, where only factors strongly associated with defaults in a 5 year horizon (1:rbc, 10: rwa and 11: lgr) have been projected at the factorial plane created by the two first axes. It is now clear that all the points referring to high risk quartiles (points 13, 103,113, 14, 104, 114) are positioned in the quantrand associated with bank failure, while the points referring to low risk quartiles (points 11, 101,111, 12, 102, 112) are positioned in the quantrand associated with non-default state. Moreover, when the exposure risk class deteriorates (moving from state 1 to state 4) the points for each of the three CAMEL factors pictured come closer to the “def” point. Therefore
we can conclude that the 3 CAMEL factors we identified (1:rbc, 10: rwa and 11: lgr) contain strong forecasting power for a bank’s failure in a 5 years horizon.

**Figure 5.12.** Correspondence analysis graph for a 5 year horizon

Tables 5.7 and 5.8 (shortened version of Table 5.8 given in Appendix) summarize the results when applying the Correspondence Analysis technique on the group of 11 CAMEL risk factors at a time horizon of 16 quarters (4 years) before failure. The proportion of variance that is explained by each axis is depicted in Table 5.7. According to it, the first factorial axis accounts for 26.89% of the total variance, the second for 18.69% and the third for 11.27%, figures that are quite close to the ones observed for the 4 year horizon (compare to Table 5.4).
Table 5.7. Correspondence analysis overall explanatory power for a 4 year horizon

<table>
<thead>
<tr>
<th>Dimension</th>
<th>principal inertia</th>
<th>percent</th>
<th>cumulative percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0216472</td>
<td>26.89</td>
<td>26.89</td>
</tr>
<tr>
<td>2</td>
<td>0.0150467</td>
<td>18.69</td>
<td>45.57</td>
</tr>
<tr>
<td>3</td>
<td>0.0089291</td>
<td>11.09</td>
<td>56.66</td>
</tr>
<tr>
<td>4</td>
<td>0.0049639</td>
<td>6.17</td>
<td>62.83</td>
</tr>
<tr>
<td>5</td>
<td>0.0026103</td>
<td>3.24</td>
<td>66.07</td>
</tr>
<tr>
<td>6</td>
<td>0.001402</td>
<td>1.74</td>
<td>67.81</td>
</tr>
<tr>
<td>7</td>
<td>0.0003186</td>
<td>0.40</td>
<td>68.21</td>
</tr>
<tr>
<td>8</td>
<td>0.0002354</td>
<td>0.29</td>
<td>68.50</td>
</tr>
<tr>
<td>9</td>
<td>0.0001466</td>
<td>0.18</td>
<td>68.68</td>
</tr>
<tr>
<td>10</td>
<td>0.0000527</td>
<td>0.07</td>
<td>68.75</td>
</tr>
<tr>
<td>Total</td>
<td>0.0805141</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.8 displays the coordinates of each risk category of all 11 CAMEL factors for the 4 year horizon as well as the coordinates of the binary supplementary variable that describes whether a bank has defaulted or not 4 years after the specific CAMEL factor data were collected.

Table 5.8. Correspondence analysis coordinates for exposure class 4 and quality metrics for a 4 year horizon (shortened version of Table 5.8 given in Appendix)

<table>
<thead>
<tr>
<th>Categories</th>
<th>overall quality</th>
<th>dimension 1</th>
<th>dimension 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>coord</td>
<td>sqcorr</td>
<td>Coord</td>
</tr>
<tr>
<td>rbc</td>
<td>0.537</td>
<td>-0.807</td>
<td>0.101</td>
</tr>
<tr>
<td>ncl</td>
<td>0.815</td>
<td>1.536</td>
<td>0.724</td>
</tr>
<tr>
<td>equ</td>
<td>0.554</td>
<td>-0.323</td>
<td>0.029</td>
</tr>
<tr>
<td>roa</td>
<td>0.671</td>
<td>2.359</td>
<td>0.499</td>
</tr>
<tr>
<td>cov</td>
<td>0.680</td>
<td>2.538</td>
<td>0.628</td>
</tr>
<tr>
<td>nim</td>
<td>0.687</td>
<td>0.895</td>
<td>0.201</td>
</tr>
<tr>
<td>eff</td>
<td>0.678</td>
<td>2.183</td>
<td>0.499</td>
</tr>
<tr>
<td>past30</td>
<td>0.717</td>
<td>0.973</td>
<td>0.462</td>
</tr>
<tr>
<td>nco</td>
<td>0.588</td>
<td>1.636</td>
<td>0.470</td>
</tr>
<tr>
<td>rwa</td>
<td>0.486</td>
<td>-0.789</td>
<td>0.158</td>
</tr>
<tr>
<td>lgr</td>
<td>0.715</td>
<td>-0.528</td>
<td>0.156</td>
</tr>
<tr>
<td>def</td>
<td>-1.466</td>
<td>-1.500</td>
<td></td>
</tr>
<tr>
<td>ndf</td>
<td>0.002</td>
<td>0.002</td>
<td></td>
</tr>
</tbody>
</table>

The coordinates of the defaulted banks are -1.466 in the first factorial axis and -1.500 in the second, while the respective coordinates of the non-defaulted banks are 0.002 in the first factorial axis and 0.002 in the second. The situation is pretty similar with the 5 year horizon and the point associated with the defaulted banks lies again at the lower-left quadrant of the factorial plane, and the point associated with the non-defaulted banks lies in the upper-right quadrant, very close to the origin. Therefore, a negative sign in both coordinates of an exposure class is indicative of the close relationship between the class and the bank failure. We notice
again that, out of the 11 CAMEL factors there are three whose both factorial coordinates are negative for the exposure class 4: rbc with coordinates (-0.807, -1.807), rwa with coordinates (-0.789, -1.327) and lgr with coordinates (-0.528, -0.706).

**Figure 5.13.** Correspondence analysis graph for a 5 year horizon (only factors strongly associated with defaults are displayed)

Moreover, it can be easily checked in the full version of Table 5.8 given in Appendix, that both coordinates for the exposure class 3 of these CAMEL factors are negative while the coordinates of the rest two exposure classes are both positive. The comment made on the fourth CAMEL factor--equity ratio-- which seems to have a similar behavior with rbc, rwa and lgr, is valid in this case as well: its quality of representation of the 4 exposure classes in the first
factorial axis is very low, so we avoid using the positioning (coordinates) of the classes in the factorial plane to arrive at conclusions pertaining to the proximity of them to the point representing the bank failure.

Since the situation is more or less similar to the 5 year period (see also Figure 5.14 and compare it to Figure 5.13) we may conclude that, under a 4 year horizon, the same 3 CAMEL factors that were identified there (1:rbc, 10: rwa and 11: lgr) contain strong forecasting power for a bank’s failure 16 quarters ahead.

**Figure 5.14.** Correspondence analysis graph for 4 year horizon (only factors strongly associated with defaults are displayed)
Let us now shift to the 3 years horizon before bank failure. The numerical results for this horizon are summarized in Table 5.9 and Table 5.10 (shortened version of Table 5.10 given in Appendix) while the respective graphical display is shown in Figure 5.15.

**Table 5.9.** Correspondence analysis overall explanatory power for a 3 year horizon

<table>
<thead>
<tr>
<th>Dimension</th>
<th>principal inertia</th>
<th>percent</th>
<th>cumulative percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0240374</td>
<td>29.46%</td>
<td>29.46%</td>
</tr>
<tr>
<td>2</td>
<td>0.0147899</td>
<td>18.13%</td>
<td>47.58%</td>
</tr>
<tr>
<td>3</td>
<td>0.0084697</td>
<td>10.38%</td>
<td>57.96%</td>
</tr>
<tr>
<td>4</td>
<td>0.0047922</td>
<td>5.87%</td>
<td>63.84%</td>
</tr>
<tr>
<td>5</td>
<td>0.0025163</td>
<td>3.08%</td>
<td>66.92%</td>
</tr>
<tr>
<td>6</td>
<td>0.0013198</td>
<td>1.62%</td>
<td>68.54%</td>
</tr>
<tr>
<td>7</td>
<td>0.0002983</td>
<td>0.37%</td>
<td>68.90%</td>
</tr>
<tr>
<td>8</td>
<td>0.0002143</td>
<td>0.26%</td>
<td>69.17%</td>
</tr>
<tr>
<td>9</td>
<td>0.0001523</td>
<td>0.19%</td>
<td>69.35%</td>
</tr>
<tr>
<td>10</td>
<td>0.0000407</td>
<td>0.05%</td>
<td>69.40%</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>0.0815974</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

First of all, a slight improvement is observed on the proportion of variance that is explained by the first axis, as well by the factorial plane formed by the first and second axis. As seen in Table 5.9 the first factorial axis accounts now for 29.46% of the total variance, the second for 18.13% and the third for 10.38%. The plane determined by the first two factorial axes, explains 47.58% of the variance in our data.

However, what seems to be quite important is that a significant change occurs in the structure of the coordinates of the risk categories of the 11 CAMEL factors for the 3 year horizon as compared to the coordinates of the binary supplementary variable that describes bank failure/non-failure. As seen in Table 5.10 and Figure 5.15, the point representing the defaulted banks has now moved to the lower-right quadrant of the factorial plane, while the point associated with the non-defaulted banks is in the upper-left quadrant. More specifically, the coordinates of the defaulted banks are 0.938 in the first factorial axis and -1.814 in the second, while the respective coordinates of the non-defaulted banks are -0.001 in the first factorial axis and 0.003 in the second.

Looking at the coordinates of the exposure classes of the 11 CAMEL factors, we can easily identify five factors whose first coordinate for the risk exposure class 4 is positive and large in volume: ncl with respective coordinates (1.583, 0.017), roa with coordinates (2.479, 0.323), cov
with coordinates (2.665, -0.304), eff with coordinates (2.257, 0.417) and nco with coordinates (1.706,-0.502).

<table>
<thead>
<tr>
<th>Categories</th>
<th>overall quality</th>
<th>dimension 1</th>
<th>dimension 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>coord</td>
<td>sqcorr</td>
</tr>
<tr>
<td>rbc</td>
<td>0.532</td>
<td>-0.306</td>
<td>0.017</td>
</tr>
<tr>
<td>ncl</td>
<td>0.826</td>
<td>1.583</td>
<td>0.769</td>
</tr>
<tr>
<td>equ</td>
<td>0.542</td>
<td>-0.067</td>
<td>0.001</td>
</tr>
<tr>
<td>roa</td>
<td>0.696</td>
<td>2.479</td>
<td>0.578</td>
</tr>
<tr>
<td>cov</td>
<td>0.715</td>
<td>2.665</td>
<td>0.701</td>
</tr>
<tr>
<td>nim</td>
<td>0.679</td>
<td>0.914</td>
<td>0.242</td>
</tr>
<tr>
<td>eff</td>
<td>0.700</td>
<td>2.257</td>
<td>0.567</td>
</tr>
<tr>
<td>past30</td>
<td>0.719</td>
<td>0.990</td>
<td>0.520</td>
</tr>
<tr>
<td>nco</td>
<td>0.614</td>
<td>1.706</td>
<td>0.554</td>
</tr>
<tr>
<td>rwa</td>
<td>0.447</td>
<td>-0.268</td>
<td>0.022</td>
</tr>
<tr>
<td>lgr</td>
<td>0.709</td>
<td>-0.313</td>
<td>0.063</td>
</tr>
<tr>
<td>def</td>
<td>0.938</td>
<td>-1.814</td>
<td></td>
</tr>
<tr>
<td>ncf</td>
<td>-0.001</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In addition, for all of them the quality of representation in the first axis is quite good (76.9%, 57.8%, 70.1%, 56.7% and 55.4% respectively). As far as the overall quality of representation is concerned, it is very good for the two of them (76.9% for ncl and 70.7% for cov) and a little inferior for the other three (58.4% for roa, 57.9% for eff and 58.3% for nco). Since the second coordinate of eff is in high contrast to the second coordinate of the point representing the defaulted banks, we may conclude that this CAMEL factor is not very informative for bank failure. Note that the second coordinate of cov owns the same sign as the defaulted banks’ respective coordinate (negative), while ncl and roa have opposite signs, however their volume and quality of representation (on the second factorial axis) is not good enough to take this fact into account.

The above observations justify that, in a 3 year horizon there is a structural change on the group of CAMEL factors, since four new factors with strong forecasting power on bank failure 12 quarters ahead, kick in: ncl, roa, cov and nco. Based on the total quality of representation and its decomposition in the first two axes, the safest choices as predictors are ncl (76.9%=76.9%+0.0%), roa (58.4%=57.8+0.06%) and cov (70.7%=70.1% +0.06%), although nco (58.4%=55.4%+3%) seems a reasonable choice as well.
Note that, at his time horizon, the exposure class 4 of the three factors selected in the 5 and 4 years horizons, do not have good qualities of representations in the first factorial and therefore their predicting power at a 3 year horizon cannot be assessed by their positioning in this axis. However, they do have adequate qualities of representations in the second factorial axis, and the sign (and volume) of their coordinates in this axis indicate closeness to the defaulted bank point. This fact is further strengthened by looking at the third factorial axis (Table 5.10 in Appendix) where the situation is similar as the one observed in axis 2.

**Figure 5.15.** Correspondence analysis graph for 3 year horizon

Therefore their predicting power at a 3 year horizon should not be disregarded i.e. one may argue that these factors do not stop carrying useful information on a bank’s failure in a 3 years horizon. We shall further elucidate on that in Section 5.3.3.
Figure 5.15 displays graphically the points associated with each category of every CAMEL factor on the plane created by the two first axes as formed in a 3 years horizon. In order to make the picture conveyed by Figure 5.15 more clear, we have created Figure 5.16, where only ncl (number 2), roa (number 4), cov (number 5) and nco (number 9) have been projected at the factorial plane.

**Figure 5.16.** Correspondence analysis graph for 3 year horizon (only factors strongly associated with defaults are displayed)

It is clear that all the points referring to high risk category of the four factors (points 24, 44, 54, 94) are positioned at the same direction in the first axis (positive) as the point associated with bank failure, while the points referring to low risk categories (points 21, 41, 51, 91 and 22, 42,
52, 92) are positioned in the opposite direction. Category 3 of all risk factors has been placed in the same direction (negative) of the second axis (see points 23, 43, 53, 93) as the point associated with bank failure. Notably, category 3 of nco (point 93) has been projected at opposite direction (in terms of first axis) as compared to the direction where the defaulted banks have taken place. This is a fact that reinforces our concerns about this factor’s capacity to carry our reliable forecast for bank failure at the time horizon under investigation (3 years).

Let us now examine the 2 years horizon (8 quarters) before bank failure by the aid of Correspondence Analysis technique. The proportion of variance that is explained by each axis in this case is depicted in Table 5.11. The first factorial axis accounts for 36.04% of the total variance, the second for 16.09% and the third for 9.18%. The plane determined by the first two factorial axes, explains 52.13% of the variance in our data. As compared to the previous horizons, a good improvement is observed on the proportion of variance that is explained by the first axis, as well by the factorial plane formed by the first and second axis (an increase more than 5% in both cases).

<table>
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<th>percent</th>
<th>cumulative percent</th>
</tr>
</thead>
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<td>36.04</td>
</tr>
<tr>
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</tr>
<tr>
<td>3</td>
<td>.0079535</td>
<td>9.18</td>
<td>61.30</td>
</tr>
<tr>
<td>4</td>
<td>.0045465</td>
<td>5.24</td>
<td>66.55</td>
</tr>
<tr>
<td>5</td>
<td>.0024288</td>
<td>2.80</td>
<td>69.35</td>
</tr>
<tr>
<td>6</td>
<td>.0011533</td>
<td>1.33</td>
<td>70.68</td>
</tr>
<tr>
<td>7</td>
<td>.0002978</td>
<td>0.34</td>
<td>71.02</td>
</tr>
<tr>
<td>8</td>
<td>.0001801</td>
<td>0.21</td>
<td>71.23</td>
</tr>
<tr>
<td>9</td>
<td>.0001469</td>
<td>0.17</td>
<td>71.40</td>
</tr>
<tr>
<td>10</td>
<td>.0000231</td>
<td>0.03</td>
<td>71.43</td>
</tr>
<tr>
<td>Total</td>
<td>.0866841</td>
<td>100.00</td>
<td></td>
</tr>
</tbody>
</table>

On inspecting Table 5.12 which displays the coordinates of each risk category of all 11 CAMEL factors, 2 years before failure, and the respective graph depicted in Figure 5.17, we see that the point associated with the defaulted banks is in the lower-left quadrant of the factorial plane, while the point associated with the non-defaulted banks is in the upper-right quadrant. More specifically, the coordinates of the defaulted banks are -3.167 in the first factorial axis and -1.807 in the second, while the respective coordinates of the non-defaulted banks are 0.005 in the first factorial axis and 0.003 in the second.
Note that, as compared to the 5 and 4 year horizons, the distance between the default and non-default representation point is larger, which in fact signifies that a better discrimination between the two states (failure/non-failure) will now be feasible by the CAMEL risk factors.

Table 5.12. Correspondence analysis coordinates for exposure class 4 and quality metrics for a 2 year horizon (shortened version of Table 5.12 given in Appendix)

<table>
<thead>
<tr>
<th>Categories</th>
<th>overall quality</th>
<th>dimension 1</th>
<th>dimension 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>coord</td>
<td>sqcorr</td>
<td>coord</td>
</tr>
<tr>
<td>rbc</td>
<td>0.529</td>
<td>0.027</td>
<td>0.000</td>
</tr>
<tr>
<td>ncl</td>
<td>0.858</td>
<td>-1.667</td>
<td>0.835</td>
</tr>
<tr>
<td>equ</td>
<td>0.515</td>
<td>-0.067</td>
<td>0.002</td>
</tr>
<tr>
<td>roa</td>
<td>0.743</td>
<td>-2.496</td>
<td>0.677</td>
</tr>
<tr>
<td>cov</td>
<td>0.761</td>
<td>-2.646</td>
<td>0.753</td>
</tr>
<tr>
<td>nim</td>
<td>0.686</td>
<td>-1.001</td>
<td>0.365</td>
</tr>
<tr>
<td>eff</td>
<td>0.742</td>
<td>-2.193</td>
<td>0.655</td>
</tr>
<tr>
<td>past30</td>
<td>0.732</td>
<td>-0.949</td>
<td>0.598</td>
</tr>
<tr>
<td>nco</td>
<td>0.671</td>
<td>-1.705</td>
<td>0.646</td>
</tr>
<tr>
<td>rwa</td>
<td>0.453</td>
<td>-0.008</td>
<td>0.000</td>
</tr>
<tr>
<td>lgr</td>
<td>0.720</td>
<td>0.344</td>
<td>0.098</td>
</tr>
<tr>
<td>def</td>
<td>-3.167</td>
<td>-1.807</td>
<td></td>
</tr>
<tr>
<td>ndf</td>
<td>0.005</td>
<td>0.003</td>
<td></td>
</tr>
</tbody>
</table>

A first observation on the first axis CAMEL risk factor coordinates provided by Table 5.12, reveals for that the majority of factors (9 out of 11) the sign of risk exposure class 4 is in compliance with the sign of the respective coordinate of the defaulted banks i.e. negative. In most cases the sign of the second coordinate is negative as well, which means that the projection of the exposure class 4 in the respective CAMEL factor lies in the lower-left quadrant of the factorial plane, that is, in the same quadrant as the defaulted banks’ projection. This assertion is not valid for ncl (whose second coordinate equals 0.116 with respective quality of representation: 0.002), roa (second coordinate: 0.425, quality of representation:), nim (second coordinate: 1.062, quality of representation: 0.184) and eff (second coordinate: 0.574, quality of representation: 0.020), however, with the exception of nim the quality of representation in the second factorial axis is negligible and the position with regards to the second axis cannot be considered as indication of association with the supplementary variable describing the defaults.

The overall conclusion drawn from the foregone analysis is that, at the 2 years horizon, almost all the risk factors are strongly associated to bank default, in the sense that a bank belonging to the risk exposure class 4 of a factor is more susceptible to bank failure. The only
exception to this rule is lgr whose coordinates in the risk exposure class 4 (0.344 and -0.731) place it far away from the point characterizing bank failure. A point boosting this assertion comes from the fact that for all CAMEL risk factors, the coordinates of the risk exposure class 1 are both positive (see Table 5.18 in Appendix), a fact indicating that this class is highly associated with non-default.

A graphical illustration of the situation described above is offered by Figures 5.17-5.18.

**Figure 5.17.** Correspondence analysis graph for 2 year horizon

![MCA coordinate plot](image-url)

Figure 5.17 displays graphically on the plane created by the two first axes (as formed for a 2 year horizon), the points associated with each category of every CAMEL factor as well as the binary supplementary variable. The point depicting a bank failure has been projected at the lower-left quadrant of the factorial plane, therefore this is the part of the plane that characterizes
bankruptcy, while the opposite part (upper-right quadrant) indicates that a bank will not fail (at the horizon under inspection).

Since the plotted points are not easy to distinguish in Figure 5.17 we produced Figures 5.18 and 5.19 where a more clear picture of the behavior of each CAMEL factor is deduced. In Figure 5.18 only the 5 factors with the strongest association to defaults have been projected to the factorial plane (factors 2, 4, 5, 7 and 9) while in Figure 5.19 the rest 5 factors are displayed (factors 1, 3, 6, 8, 10). The remaining factor (11:lgr) has not been projected at all, since it appears to be non-significant in predicting bank failure.

A simple observation of Figures 5.18 and 5.19 reveals that the exposure classes 3 and 4 for all significant CAMEL factors are placed in the positive side of the factorial axes (when treating each one separately, or for both of them), while exposure classes 1 and 2 have a positive direction in either axis, or in both. Therefore, at a 2 year time horizon, all CAMEL factors, with the notable exception of lgr, contain strong forecasting power for a bank’s failure.

**Figure 5.18.** Correspondence analysis graph for 2 year horizon (only the 5 factors with the strongest association to defaults are displayed)
**Figure 5.19.** Correspondence analysis graph for 2 years horizon (only the rest 5 factors strongly associated with defaults are displayed)

The numerical results for a 1 year horizon are summarized in Table 5.13 and Table 5.14 (for the full version of Table 5.14 see Appendix) while the respective graphical display is shown in Figures 5.20-5.22. In this case a substantial improvement is noted on the proportion of variance that is explained by the first axis (it becomes 41.59%). The plane determined by the first two factorial axes, explains 55.92% of the variance in our data. The structure of the coordinates shown in Table 5.14, and further illustrated in Figures 5.20-5.22 is quite similar to the structure discovered in the 2 years horizon, so no further comment will be added here.

| Table 5.13. Correspondence analysis overall explanatory power for 1 year horizon |
|-----------------------------|----------------|--------------|----------------|
| Dimension | principal inertia | percent | cumulative percent |
| 1 | .0385048 | 41.59 | 41.59 |
| 2 | .0132683 | 14.33 | 55.92 |
| 3 | .0078762 | 8.51 | 64.42 |
| 4 | .0043055 | 4.65 | 69.07 |
| 5 | .0023746 | 2.56 | 71.64 |
| 6 | .0009861 | 1.07 | 72.70 |
| 7 | .0003207 | 0.35 | 73.05 |
| 8 | .0001764 | 0.19 | 73.24 |
| 9 | .000138 | 0.15 | 73.39 |
| 10 | .0000158 | 0.02 | 73.41 |
| Total | .0925887 | | |
Table 5.14. Correspondence analysis coordinates for exposure class 4 and quality metrics for 1 year horizon (shortened version of Table 5.14 given in Appendix)

<table>
<thead>
<tr>
<th>Categories</th>
<th>overall quality</th>
<th>dimension 1</th>
<th>dimension 2</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>coord</td>
<td>sqcorr</td>
<td>Coord</td>
</tr>
<tr>
<td>rbc</td>
<td>0.530</td>
<td>-0.077</td>
<td>0.002</td>
</tr>
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<td>ncl</td>
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<td>0.870</td>
</tr>
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<td>equ</td>
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</tr>
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<tr>
<td>cov</td>
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<td>-2.600</td>
<td>0.779</td>
</tr>
<tr>
<td>nim</td>
<td>0.698</td>
<td>-1.014</td>
<td>0.445</td>
</tr>
<tr>
<td>eff</td>
<td>0.777</td>
<td>-2.130</td>
<td>0.716</td>
</tr>
<tr>
<td>past30</td>
<td>0.751</td>
<td>-0.905</td>
<td>0.664</td>
</tr>
<tr>
<td>nco</td>
<td>0.721</td>
<td>-1.703</td>
<td>0.710</td>
</tr>
<tr>
<td>rwa</td>
<td>0.477</td>
<td>-0.034</td>
<td>0.001</td>
</tr>
<tr>
<td>lgr</td>
<td>0.743</td>
<td>0.455</td>
<td>0.187</td>
</tr>
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<td>def</td>
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<td>-5.090</td>
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</tr>
<tr>
<td>ndf</td>
<td></td>
<td>0.007</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5.20. Correspondence analysis graph for 1 year horizon
Figure 5.21. Correspondence analysis graph for 2 year horizon (only the 5 factors with the strongest association to defaults are displayed)

Figure 5.22. Correspondence analysis graph for 2 years horizon (only the rest 5 factors strongly associated with defaults are displayed)
Exploiting the findings of the present section and Section 5.3.2, we may state that, according to the group of CAMEL factors that send out signals for bank failure, we may think of breaking up the bank life cycle (prior to failure) to the following 3 stages:

**STAGE I:** 4-5 years prior to failure where rbc, lg and rwa, carry useful information for failure.

**STAGE II:** 3 years prior to failure where a new set of significant predictors kick in: ncl, roa, cov and probably nco. The strength of the factors identified in Stage I, seem to keep carrying useful information on bank failure, for a 3 year horizon as well.

**STAGE III:** It starts about 2 years prior to failure when signals are received from all CAMEL factors except lgr, which, at this stage does not seem carrying useful information about the approaching failure.

Although the Correspondence Analysis approach has identified three Stages for the bank life cycle prior to failure, there is still an open question about the significance of Stage I associated factors when Stage II commences (recall that their quality of representation in the first factorial axe is not high and we their predicting power at a 3 year horizon is disputable; the evidence provided by their positioning in the second and third factorial axis – which however do not have the power contained in the first axis - was exploited so as these factors not be disregarded). Moreover no quantification of the significance of the factors has been accomplished in the sense that we have not developed any model facilitating the task of assessing the probability of default of a bank for each exposure class, along the 3 stages. The answer to the question stated before and the task of assessing the default probabilities of default will be completed later on in Sections 5.3.3 and 5.3.4 by the aid of alternative statistical techniques.

### 5.3.3. Logistic Regression Models for Bank Failures

In the next section we shall construct a set of logit (logistic) models where the dependent variable is an indicator denoting bank failure at time $t$, more specifically

$$D_{it} = \begin{cases} 
1, & \text{if bank } i \text{ failed at time } t \\
0, & \text{otherwise.}
\end{cases}$$ (5.1)
When the time $t$ is conferred from the context we shall be suppressing the index $t$ in the notation. Likewise we may suppress the index $i$, and simply use the dummy variable $D$ to describe the status of a specific bank at a specific time point.

Let us begin by introducing the necessary theory and notation for the logistic regression model. Logistic regression is a statistical tool that can be used to describe the relationship of several predictors, $X_1, X_2, ..., X_k$ to a dichotomous dependent variable, such as the dummy variable $D$ defined above. Other modeling approaches are possible also, but logistic regression is by far the most popular technique for analyzing financial data related to dichotomous variables such as bank failure/non failure, delay/non delay in loans payments etc.

The mathematical modelling of the logistic regression formula arises from the logistic function, which is pictured in Figure 5.23. This function is given by

$$f(z) = \frac{e^z}{1 + e^z}$$

(5.2)

The fact that the logistic function $f(z)$ ranges between 0 and 1 is the primary reason it is used in the logistic regression model which aims at describing a probability, which is always bounded by 0 and 1. In our context such a probability denotes the risk of an individual bank to fail (default). The use of the logistic function, ensures that whatever estimate of risk we get, it will always be between 0 and 1.

Now, let us move from the logistic function to the logistic regression model. To achieve that, we simply write the argument $z$ as a linear combination of several explanatory variables (predictors), say $X_1, X_2, ..., X_k$. Thus we have

$$z = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_k X_k$$

(5.3)

and $f(z)$ becomes
In financial terms, $X_1, X_2, ..., X_k$ may describe factors that affect the status of a bank (failure/no failure) (e.g. the CAMEL factors mentioned in the previous sections) and we wish to explore whether these factors affect the probability that the bank will default within a specific time horizon. The probability being modeled can be denoted by the conditional probability $Pr[D=1|X_1, X_2, ..., X_k]$. It is worth noting that, since $D$ is a binary random variable, the aforementioned conditional probability can also be viewed as a conditional expectation, namely

$$1 \cdot Pr[D=1|X_1, X_2, ..., X_k] + 0 \cdot Pr[D=0|X_1, X_2, ..., X_k] = E[D|X_1, X_2, ..., X_k]$$  \hfill (5.5)

The logistic model assumes that the expression for the default probability is given by the expression

$$Pr[D=1|X_1, X_2, ..., X_k] = \frac{1}{1 + e^{-(\beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_k X_k)}}$$  \hfill (5.6)

Combining the last two formulae, we may write

$$E[D|X_1, X_2, ..., X_k] = \frac{1}{1 + e^{-(\beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_k X_k)}}$$  \hfill (5.7)

or equivalently

$$D = \frac{1}{1 + e^{-(\beta_0 + \beta_1 X_1 + \beta_2 X_2 + ... + \beta_k X_k)}} + u$$  \hfill (5.8)

where $u$ is an error term with mean 0. The last formula, if we bring back the (suppressed) index $i$ (which refers to the individual banks from which the exploratory characteristics/factors $X_1, X_2, ..., X_k$ were gathered), takes on the form

$$D_i = \frac{1}{1 + e^{-(\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + ... + \beta_k X_{ki})}} + u_i \quad i = 1, 2, ....$$  \hfill (5.9)

The terms $\beta_0, \beta_1, ..., \beta_k$ in this model represent unknown parameters that need to be estimated from the data obtained on the $X$'s and on $D$ (bank defaulted/did not default) for a group of banks. Once the estimation procedure is carried out, we can use the last formula to obtain the probability that an individual bank could fail over some period, by simply inserting the values of the factors in (5.1).
In the sequel, we shall use the simpler notation \( p(X_1, X_2, \ldots, X_k) = p(X) \) for the conditional probability shown in (5.1), i.e.

\[
p(X) = p(X_1, X_2, \ldots, X_k) = \Pr[D = 1 \mid X_1, X_2, \ldots, X_k] = \frac{1}{1 + e^{-(\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k)}}
\]  

(5.10)

The logistic regression model can be equivalently written in an alternative formula, which is called the logit form of the model. To get the logit from the logistic model, we proceed to a transformation as follows:

\[
p(X) = \frac{1}{1 + e^{-(\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k)}} \iff \frac{1}{p(X)} = 1 + e^{-(\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k)} \iff \log \frac{1}{p(X)} = -(\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k)
\]  

(5.11)

which in turn may be written as follows

\[
\log \frac{p(X)}{1 - p(X)} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k
\]  

(5.13)

The left hand side of (5.4) is usually called “the logit transformation of \( p(X) \)”, and is denoted as \( \text{logit}(p(X)) \). Expression (5.4) allows the computation of \( \text{logit}(p(X)) \), for an individual (e.g. a bank) with predictor variables given by \( X = (X_1, X_2, \ldots, X_k) \). Note that, when a logistic regression model is assumed, the logit transformation of the conditional probability \( \Pr[D = 1 \mid X_1, X_2, \ldots, X_k] = p(X) \) simplifies to the linear combination of the \( X_1, X_2, \ldots, X_k \).

Another interesting observation with respect to the logistic model is that the quantity \( p(X)/(1 - p(X)) \) appearing in (5.4) is connected to the odds of the event “bank failure” for a bank with independent variables (predictors) specified by \( X = (X_1, X_2, \ldots, X_k) \). We recall that the odds show the ratio of the probability that some event will occur over the probability that the same event will not occur (or equivalently it's complementar will occur). Therefore, an odds is of the form \( p/(1 - p) \), where \( p \) denotes the probability of the event of interest, a fact that makes clear that, in the logistic model, the logarithm of the odds is a linear combination of the predicted variables. An odds value of 4 (respectively 1/4) can be interpreted to mean that the probability of the event occurring is 4 times (respectively 1/4 times) the probability of the event.
not occurring. Alternatively, we can state that the odds are 4 to 1 (respectively 1 to 4) that the event will take place.

Let us next move to the interpretation of the parameters of a logistic (or equivalently a logit) model. Apparently, if we plug in 0 for all \( X_1, X_2, \ldots, X_k \) in the formula (6.4), we find that the logit of \( p(X) \) reduces to \( \beta_0 \). Hence \( \beta_0 \) may be interpreted as the log odds for a bank with zero values for all \( X \)'s. Since the condition “all \( X_1, X_2, \ldots, X_k \) have obtained the value 0” may be meaningless in many cases (e.g. the CAMEL factors are all zero, is rather unrealistic), we prefer referring to \( \beta_0 \) as the log of the baseline odds.

The interpretation of the remaining parameters \( \beta_1, \beta_2, \ldots, \beta_k \) involves the notion of odds ratio. Let us start by examining what happens in model (5.4) when only one of the \( X \)'s varies while keeping the others fixed. Assume for example that \( X_1 \) increases by 1 (it becomes \( X_1 + 1 \)) while \( X_1, X_2, \ldots, X_k \) remain unaltered. Then, in view of (6.4) we may write

\[
\text{logit} (p(X)) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_k X_k
\]

and

\[
\text{logit} (p(X')) = \beta_0 + \beta_1 (X_1 + 1) + \beta_2 X_2 + \ldots + \beta_k X_k
\]

where \( X' = (X_1 + 1, X_2, \ldots, X_k) \). It is now evident that

\[
\beta_i = \text{logit} (p(X')) - \text{logit} (p(X))
\]

that is \( \beta_i \) represents the change in the log odds that would result from a one unit change in the variable \( X_1 \) when \( X_1, X_2, \ldots, X_k \) are fixed. In general, \( \beta_j \) will represent the change in the log odds that would result from a one unit change in the variable \( X_j \) when all other \( X \)'s are fixed.

A more appealing interpretation comes out if we replace the logits in the last formula by

\[
\text{logit} (p(X)) = \log \frac{p(X)}{1 - p(X)}, \quad \text{logit} (p(X')) = \log \frac{p(X')}{1 - p(X')}
\]

darof arriving at the expression
Recalling that the \( p(X)/(1 - p(X)) \) is the odds of the event “bank failure” for a bank with independent variables (predictors) specified by \( X = (X_1, X_2, ..., X_k) \) and \( p(X')/(1 - p(X')) \) is the odds of the same event for a bank with specified by \( X' = (X_1 + 1, X_2, ..., X_k) \) we realize that \( \beta_1 \) is the logarithm of the odds ratio for bank failure when comparing (the odds of) two banks with a one unit difference in the variable \( X_1 \) and the rest variables \( X_1, X_2, ..., X_k \) fixed. Clearly (3.3.4.6) can be expressed in the equivalent form

\[
e^{\beta_1} = \frac{p(X)}{1 - p(X)} \frac{1 - p(X)}{p(X')}
\]

which states that the odds ratio for bank failure when comparing (the odds of) two banks with a one unit difference in the variable \( X_1 \) and the rest variables \( X_1, X_2, ..., X_k \) fixed, is given by \( e^{\beta_1} \).

These results can be easily generalized for any of the parameters \( \beta_j, j = 1,2, ..., k \).

In the case where the predictor variable \( X_1 \) is dichotomous (and is coded as 0-1 dummy variable), the shift from the specification \( X = (X_1, X_2, ..., X_k) = (0, X_2, ..., X_k) \) to \( X' = (X_1 + 1, X_2, ..., X_k) = (1, X_2, ..., X_k) \) means that we compare two banks that differ in the state of the characteristic described by \( X_1 \). For example, if \( X_1 \) describes different buckets (categories) on a CAMEL factor, we are comparing banks which belong two different buckets of this CAMEL factor, and have identical specifications on the rest factors.

The logistic model, as described in the financial framework set-up (prediction model for bank failure), is defined with a follow-up study orientation. More specifically, it describes the default probability of a bank, expressed as a function of independent variables (predictors) \( X_1, X_2, ..., X_k \) which have been measured at the start of a fixed follow-up period. Therefore one
may raise the question whether it is correct to use this model for cross-sectional studies, as the one analyzed in the present chapter. Fortunately, the answer is positive, and can be based on the reports of the articles of Prentice and Pike (1979) and Breslow and Day (1981), where specific "robust" conditions have been identified, under which the logistic model can be used with case-control data. The outcomes of these papers carry over nicely to cross-sectional studies as well, though this has not been explicitly demonstrated in the literature. As a consequence, even with a cross-sectional design, one can pretend, when doing the analysis, that the dependent variable is the binary variable describing the bank status at the end of the study (default/non default) and the independent variables are exposure status and any other covariates of interest. When using a logistic model with cross-sectional data, you can treat the data as if it came from a follow-up study, and still obtain valid results.

Although logistic models can be applied to cross-sectional studies, there is one important limitation in the interpretation of the statistical outcomes: while in follow-up studies a fitted logistic model can be used to predict the risk (e.g. default probability) for an individual with specified specification on the independent variables $X_1, X_2, ..., X_k$, the model cannot be used to predict individual risk for cross-sectional studies. In the latter case, only estimates of odds ratios can be obtained and exploited for drawing conclusions on the physical problem at hand.

The mathematical explanation for this problem is that predicted risks are obtained from the formula (c.f. (5.5))

$$
\log \frac{\hat{p}(X)}{1 - \hat{p}(X)} = \hat{\beta}_0 + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 + ... + \hat{\beta}_k X_k
$$

(5.20)

Which requires valid estimates for $\beta_1, ..., \beta_k$ and $\beta_0$. When using cross-sectional data, the parameter $\beta_0$ cannot be estimated without knowing the sampling fraction of the population, as a consequence we cannot establish a valid estimate of the predicted risk $\hat{p}(X)$. In follow-up studies, this problem does not exist, so $\beta_0$ can be safely estimated, and then an estimate $\hat{p}(X)$ of $p(X)$ could be calculated.

In cross-sectional studies, although $\beta_0$ cannot be estimated, the rest parameters $\beta_1, ..., \beta_k$ can be estimated. On the other hand, as noted earlier (see the discussion after formula (6.4)), the
parameters provide enough information about odds ratios, namely $e^{\beta_j}$, $j=1,2,\ldots,k$, is the odds ratio associated to the predictor variable $X_j$, and can be naturally estimated by $e^{\hat{\beta}_j}$. Therefore, although we cannot estimate $\beta_0$ in a cross-sectional study, it is still feasible to obtain valid estimates of odds ratios.


### 5.3.4. Assessing Forecasting Performance Across Horizons

As stated in Section 5.3.1, $F_j$ denotes the $j$-th risk factor ($j=1,2,\ldots,11$) from the family of CAMEL factors and for each risk factor $F_j$ we use the observed quartiles to construct four exposure levels $E_j^r$. The index $j$ indicates the risk factor and $r$ ($r=1,2,3,4$) denotes the exposure level number. The cut-points (quartiles) for the each exposure class and Camel factors in 5 different time horizons, are depicted in Table 5.2.

In order to describe a bank’s exposure to each particular risk factor at time $t$, we shall use four dummy variables defined as follows:

$$C_{i,j,t}^{(r)} = \begin{cases} 
1, & \text{if bank } i \text{ belongs, at time } t, \text{ to the } r\text{-th risk exposure class of risk factor } j \\
0, & \text{otherwise}
\end{cases} \quad (5.21)$$

for $r=1,2,3,4$ and $j=1,2,\ldots,11$.

A different logit model will be established for each risk factor in order to assess its forecasting ability. The model will be applied for five different horizons prior to failure: 5 years (20 quarters), 4 years (16 quarters), 3 years (12 quarters), 2 years (8 quarters), and 1 year (4 quarters).
The explanatory variables are the three risk exposure classes, having the lowest risk exposure class as a reference group. As dependent variable we shall use the indicator (dummy) variable

\[ D_{i,t} = \begin{cases} 1, & \text{if bank } i \text{ failed at time } t \\ 0, & \text{otherwise,} \end{cases} \quad (5.22) \]

which describes the status of the \( i \)-th bank (defaulted/non-defaulted) at time \( t \). In view of (5.1), we may write the logistic regression model as follows.

\[ D_{i,t} = \frac{1}{1 + \exp[-(a + \beta_2 C_{i,j,t-k}^{(2)} + \beta_3 C_{i,j,t-k}^{(3)} + \beta_4 C_{i,j,t-k}^{(4)}) + u_{i,t}]} \quad (5.23) \]

where \( k = 1,2,3,4,5 \) (years) indicates the horizon checked each time and \( u_{i,t} \) the error term of the model.

Letting \( p_{i,t} = E(D_{i,t}) = \Pr(D_{i,t} = 1) \) denote the default probability of bank \( i \) at time \( t \), the above model takes on the form

\[ p_{i,t} = E(D_{i,t}) = \Pr(D_{i,t} = 1) = \frac{1}{1 + \exp[-(a + \beta_2 C_{i,j,t-k}^{(2)} + \beta_3 C_{i,j,t-k}^{(3)} + \beta_4 C_{i,j,t-k}^{(4)})]} \quad (5.24) \]

or equivalently

\[ \log \left( \frac{p_{i,t}}{1 - p_{i,t}} \right) = a + \beta_2 C_{i,j,t-k}^{(2)} + \beta_3 C_{i,j,t-k}^{(3)} + \beta_4 C_{i,j,t-k}^{(4)}. \quad (5.25) \]

This is a typical logistic regression model with three binary predictors \( C_{i,j,t-k}^{(2)}, C_{i,j,t-k}^{(3)}, C_{i,j,t-k}^{(4)} \)

<table>
<thead>
<tr>
<th>Table 5.15.</th>
<th>Value of the logit ( p_{i,t} ) in terms of the model parameters for the 4 exposure classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>Exposure level</td>
</tr>
<tr>
<td>1</td>
<td>( E_{1j} )</td>
</tr>
<tr>
<td>2</td>
<td>( E_{2j} )</td>
</tr>
<tr>
<td>3</td>
<td>( E_{3j} )</td>
</tr>
<tr>
<td>4</td>
<td>( E_{4j} )</td>
</tr>
</tbody>
</table>
Apparently, the right hand side of (3.3.4.3) is the logit transformation of \( p_{i,t} = E(D_{i,t}) = \Pr(D_{i,t} = 1) \). Table 3.3.4.1 displays the value of logit( \( p_{i,t} \) ) in terms of the model parameters when a bank belongs to each of the 4 exposure classes \( E_j^1, E_j^2, E_j^3, E_j^4 \). To prevent any misunderstanding, from now on we shall refer to the four exposure classes simply by the numbers 1, 2, 3 and 4.

Denoting by \( odd_i \) the odds (with respect to failure) of a bank belonging to risk class \( i \) (\( i = 1,2,3,4 \)), it is clear from Table 5.15, that

\[
\log(odd_2) - \log(odd_1) = (a + \beta_2) - a = \beta_2
\]

(5.26)

\[
\log(odd_3) - \log(odd_1) = (a + \beta_3) - a = \beta_3
\]

(5.27)

\[
\log(odd_4) - \log(odd_1) = (a + \beta_4) - a = \beta_4
\]

(5.28)

which in turn yields

\[
\log\left(\frac{odd_2}{odd_1}\right) = \beta_2 \Leftrightarrow \exp(\beta_2) = \frac{odd_2}{odd_1}
\]

(5.29)

\[
\log\left(\frac{odd_3}{odd_1}\right) = \beta_3 \Leftrightarrow \exp(\beta_3) = \frac{odd_3}{odd_1}
\]

(5.30)

\[
\log\left(\frac{odd_4}{odd_1}\right) = \beta_4 \Leftrightarrow \exp(\beta_4) = \frac{odd_4}{odd_1}
\]

(5.31)

Therefore, the quantities \( \exp(\beta_2) \), \( \exp(\beta_3) \), \( \exp(\beta_4) \) are equal to the odds ratio of the exposure classes 2, 3, 4 respectively with respect to the exposure class 1 (baseline). This finding is similar to the comment made in the discussion following formula (5.6). Apparently, if \( \hat{\beta}_2, \hat{\beta}_3, \hat{\beta}_4 \) are the estimates of parameters \( \beta_2, \beta_3, \beta_4 \) by the aid of the available data, the odds ratios for the exposure classes 2, 3, 4 can be estimated by \( \exp(\hat{\beta}_2), \exp(\hat{\beta}_3), \exp(\hat{\beta}_4) \) respectively.

It is worth stressing that, the suggested model does not predict failure at the bank level, but rather assists us in identifying which risk exposure class is more (less) vulnerable to failure for each time horizon, as reflected in failure rates between risk exposure classes. The assessment of the comparison of each exposure class to risk class 1 (baseline) will be carried out by the use of \( \exp(\hat{\beta}_2), \exp(\hat{\beta}_3), \exp(\hat{\beta}_4) \) after the estimation procedure is completed for each time horizon.
Our interest will be focused on the following issues:

(i) decide which risk factors exhibit ex ante forecasting ability of failure and at which horizon

(ii) the quantification of differential failure risk across risk exposure classes.

According to the related literature, it is essential each factor's forecasting performance across different horizons to satisfy the following properties:

(a) **significance**, that is the risk factor should contain significant forecasting power over failure,

(b) **correct direction**, which essentially requires that the odds of failure are above unity for all risk exposure classes, in comparison to the reference class, and

(c) **monotonicity**, which essentially requires that the odds of failure should increase monotonically across risk exposure class.

In the analysis that follows it is clear that the above properties are valid for the majority of risk factors that are examined.

Table 5.16 reports the estimation results from the set of logit models across CAMEL metrics five years prior to failure. As expected, most of the risk factors (8 out of 11) do not contain any significant forecasting power. The only factors with significant power are rbc, rwa and lg. Moreover, these factors also satisfy the correct direction and monotonicity criteria. The explanatory power of factors is 2.77% (rbc), 3.2% (lg) and 5.9% (rwa).

<table>
<thead>
<tr>
<th>Risk factor</th>
<th>Exposure</th>
<th>$R^2$</th>
<th>Wald</th>
<th>Log likelihood</th>
<th>obs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rbc</td>
<td>0.881</td>
<td>0.930***</td>
<td>4.272***</td>
<td>0.0277</td>
<td>140.27***</td>
</tr>
<tr>
<td>ncl</td>
<td>0.599***</td>
<td>0.591***</td>
<td>0.002***</td>
<td>0.0041</td>
<td>21.16***</td>
</tr>
<tr>
<td>equ</td>
<td>0.764</td>
<td>1.047</td>
<td>1.271*</td>
<td>0.0021</td>
<td>10.91**</td>
</tr>
<tr>
<td>roa</td>
<td>0.942</td>
<td>0.931</td>
<td>1.115</td>
<td>0.0004</td>
<td>1.82</td>
</tr>
<tr>
<td>cov</td>
<td>0.746*</td>
<td>0.746*</td>
<td>0.806</td>
<td>0.0010</td>
<td>3.40</td>
</tr>
<tr>
<td>nim</td>
<td>0.744**</td>
<td>0.480***</td>
<td>0.465***</td>
<td>0.0069</td>
<td>35.99***</td>
</tr>
<tr>
<td>eff</td>
<td>0.740**</td>
<td>0.673**</td>
<td>0.923</td>
<td>0.0017</td>
<td>8.79**</td>
</tr>
<tr>
<td>Past30</td>
<td>0.637***</td>
<td>0.590***</td>
<td>0.503***</td>
<td>0.0049</td>
<td>24.99***</td>
</tr>
<tr>
<td>nco</td>
<td>1.275</td>
<td>0.839</td>
<td>0.839</td>
<td>0.0024</td>
<td>12.36***</td>
</tr>
<tr>
<td>Rwa</td>
<td>1.091</td>
<td>3.139***</td>
<td>10.585***</td>
<td>0.0585</td>
<td>303.33***</td>
</tr>
<tr>
<td>lgr</td>
<td>1.172</td>
<td>1.912***</td>
<td>5.217***</td>
<td>0.0320</td>
<td>152.99***</td>
</tr>
</tbody>
</table>
For banks belonging to the 3rd or 4th risk category of rbc the odds of failure are 1.93 and 4.27 times higher compared to the 1st risk category. For the 3rd and 4th risk category of lg, the odds of failure are 1.91 and 5.22 times higher. Based on the risk exposure of rwa, we find that the odds of failure for the 3rd risk category are 3.14 time higher, while for the 4th are 10.59 time higher.

Looking at the estimation results for the 4-year horizon (see Table 5.17) we find a similar picture to that for the 5-year. Essentially the only significant predictors of failure are again rbc, rwa and lgr. Moreover, the estimated failure odds ratios for the 3rd and the 4th risk exposure categories are relatively close to those for the 5-year horizon.

Looking at the 3-year horizon (see Table 5.18) we uncover that the set of significant predictors is now augmented by ncl, roa and cov.

<table>
<thead>
<tr>
<th>Table 5.17. 16 quarters forecasting horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk factor</td>
</tr>
<tr>
<td>--------------</td>
</tr>
<tr>
<td>Rbc</td>
</tr>
<tr>
<td>ncl</td>
</tr>
<tr>
<td>equ</td>
</tr>
<tr>
<td>roa</td>
</tr>
<tr>
<td>cov</td>
</tr>
<tr>
<td>nim</td>
</tr>
<tr>
<td>eff</td>
</tr>
<tr>
<td>Past30</td>
</tr>
<tr>
<td>nco</td>
</tr>
<tr>
<td>Rwa</td>
</tr>
<tr>
<td>lgr</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 5.18. 12 quarters forecasting horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk factor</td>
</tr>
<tr>
<td>--------------</td>
</tr>
<tr>
<td>Rbc</td>
</tr>
<tr>
<td>ncl</td>
</tr>
<tr>
<td>equ</td>
</tr>
<tr>
<td>roa</td>
</tr>
<tr>
<td>cov</td>
</tr>
<tr>
<td>nim</td>
</tr>
<tr>
<td>eff</td>
</tr>
<tr>
<td>Past30</td>
</tr>
<tr>
<td>nco</td>
</tr>
<tr>
<td>Rwa</td>
</tr>
<tr>
<td>lgr</td>
</tr>
</tbody>
</table>
In particular, we find that banks in the 4th risk category of ncl exhibit 1.77 times higher probability of failure. Their counterparts in the roa factor show 2.24 times higher probability of failure. The odd ratio related to rbc is comparable in magnitude to its previous level, although it is somewhat increased, whereas the odd ratio associated to the lgr risk factor seems to fade away gradually. What is impressive in this forecasting horizon is the increase in the failure signals carried by the rwa factor.

For instance, the odds of failure for banks in the 3rd and 4th risk categories are 4.09 and 23.59 times higher respectively. Moreover, the explanatory power of rwa is almost 8%, more than double compared to the explanatory power of the second best predictor (rbc).

Table 5.19. 8 quarters forecasting horizon

<table>
<thead>
<tr>
<th>Risk factor</th>
<th>Exposure</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>$R^2$</th>
<th>Wald</th>
<th>Log likelihood</th>
<th>obs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rbc</td>
<td></td>
<td>2.267***</td>
<td>** 4.870***</td>
<td><strong>16.847</strong>*</td>
<td>0.0582</td>
<td>340.30***</td>
<td>-2752.044</td>
<td>340,494</td>
</tr>
<tr>
<td>ncl</td>
<td></td>
<td>0.882</td>
<td>1.441</td>
<td>7.697***</td>
<td>0.0575</td>
<td>335.99***</td>
<td>-2755.864</td>
<td>342,014</td>
</tr>
<tr>
<td>equ</td>
<td></td>
<td>1.340</td>
<td><strong>1.888</strong>*</td>
<td><strong>2.833</strong>*</td>
<td>0.0096</td>
<td>56.04***</td>
<td>-2898.204</td>
<td>344,185</td>
</tr>
<tr>
<td>roa</td>
<td></td>
<td><strong>2.189</strong>*</td>
<td><strong>2.332</strong>*</td>
<td><strong>11.400</strong>*</td>
<td>0.0588</td>
<td><strong>353.30</strong>*</td>
<td>-2825.762</td>
<td>421,904</td>
</tr>
<tr>
<td>cov</td>
<td></td>
<td>1.064</td>
<td><strong>1.807</strong>*</td>
<td><strong>6.210</strong>*</td>
<td>0.0409</td>
<td>195.11***</td>
<td>-2285.964</td>
<td>238,743</td>
</tr>
<tr>
<td>nim</td>
<td></td>
<td>0.765</td>
<td>1.177</td>
<td>2.562***</td>
<td>0.0149</td>
<td>86.97***</td>
<td>-2882.732</td>
<td>344,175</td>
</tr>
<tr>
<td>eff</td>
<td></td>
<td>0.648***</td>
<td>0.865</td>
<td>2.544***</td>
<td>0.0198</td>
<td>115.62***</td>
<td>-2868.368</td>
<td>344,142</td>
</tr>
<tr>
<td>Past30</td>
<td></td>
<td>0.979</td>
<td><strong>1.852</strong>*</td>
<td><strong>4.136</strong>*</td>
<td>0.0244</td>
<td>140.94***</td>
<td>-2813.760</td>
<td>307,647</td>
</tr>
<tr>
<td>nco</td>
<td></td>
<td>1.262</td>
<td><strong>1.741</strong>*</td>
<td><strong>4.470</strong>*</td>
<td>0.0229</td>
<td>133.66***</td>
<td>-2856.645</td>
<td>341,664</td>
</tr>
<tr>
<td>Rwa</td>
<td></td>
<td><strong>2.125</strong>*</td>
<td><strong>11.010</strong>*</td>
<td><strong>32.722</strong>*</td>
<td>0.0732</td>
<td>428.38***</td>
<td>-2712.040</td>
<td>344,189</td>
</tr>
<tr>
<td>lgr</td>
<td></td>
<td><strong>0.679</strong>*</td>
<td><strong>0.651</strong>*</td>
<td>1.180</td>
<td>0.0041</td>
<td>23.65***</td>
<td>-2850.218</td>
<td>300,493</td>
</tr>
</tbody>
</table>

Table 5.19 summarizes the results for the 2-year horizon for which there are several interesting findings. First, lgr loses completely its forecasting ability over future failures; its predictive ability reduces to 0.4% and the signal strength of risk classes 3 and 4 are very low (as a matter of fact risk class 3 has an odds ratio less than 1). Second, signals now are coming from more factors (equ, past30, nco), suggesting that problems start to mount two years before failure, thereby identifying banks at default risk more clearly. The factors with the highest predictive ability are rbc (5.82%) and rwa (7.32%). Banks in the 4th risk group of rbc exhibit 16.85 times higher odds of failure, while their rwa counterparts 32.72, estimates which are indicative of the signal strength. The next highest signal is provided by roa according to which risk exposure class 4 banks show 11.4 times higher odds of failure.

Table 5.20 reports the estimation results from the set of logit models across CAMEL metrics
one year prior to failure. As expected, given the proximity to failure, most of the risk factors (8 out of 11) contain significant forecasting power. Note that lgr does not have any predicting power for a 1-year horizon as well, since the odds ratio of risk classes 3 and 4 are much lower than 1. The factors with the lower significant power are nim, eff, rwa and lgr. However, with the exception of lgr the signal strength of risk class 4 is quite strong.

The factors with the highest explanatory power are roa, ncl, cov and rbc which account for 13.04%, 12.94%, 12.49% and 12.21% of sample variance respectively. Focusing on these three factors we find that banks in the 4th risk exposure category of roa, exhibit 23.08 higher probability of failure, those that belong to the 4th risk exposure category of ncl exhibit 57.54 higher probability of failure, the ones in the 4th risk exposure category cov exhibit 53.24 higher probability of failure, and finally, based on rbc we observe a 56.36 times higher probability for banks in the 4th risk exposure category. The corresponding odds ratios for the 3rd risk exposure categories for ncl, cov and rbc are 2.67, 3.67 and 3.5 times respectively.

Based on the overall estimation results we may conclude that indeed there are CAMEL factors that contain significant ex ante explanatory power over future bank failures. This suggests that particular groups of banks can be identified as being ex ante more susceptible to failure with distinctly higher odds of failure.

Another important finding is that one may distinguish three stages as far as the arrivals of failure signals are concerned. As depicted in Figure 5.24 and Figure 5.25, at distant horizons (4-5 years) early failure signals can be obtained from rbc, rwa and lgr

<table>
<thead>
<tr>
<th>Table 5.20. 4 quarters forecasting horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk factor</td>
</tr>
<tr>
<td>-------------</td>
</tr>
<tr>
<td>Rbc</td>
</tr>
<tr>
<td>ncl</td>
</tr>
<tr>
<td>equ</td>
</tr>
<tr>
<td>roa</td>
</tr>
<tr>
<td>cov</td>
</tr>
<tr>
<td>nim</td>
</tr>
<tr>
<td>eff</td>
</tr>
<tr>
<td>Past30</td>
</tr>
<tr>
<td>nco</td>
</tr>
<tr>
<td>Rwa</td>
</tr>
<tr>
<td>lgr</td>
</tr>
</tbody>
</table>
As we move closer to failure time, signals are conveyed by 3 more CAMEL factors. For instance, 3 years prior to failure ncl, roa and cov kick in (see Figure 5.26).

**Figure 5.24.** Failure signals from significant factors in 5 year horizon

![5 year horizon: odds ratios for exposure classes 3 and 4](image)

**Notes:** On the horizontal axis 3 and 4 stand for the 3rd and 4th risk exposure class respectively

**Figure 5.25.** Failure signals from significant factors in 4 year horizon

![4 year horizon: odds ratios for exposure classes 3 and 4](image)

**Notes:** On the horizontal axis 3 and 4 stand for the 3rd and 4th risk exposure class respectively
Finally, as seen in Figure 5.27, two years prior to failure the set of CAMEL factors conveying a failure signal is augmented by equity, nim, eff, Past30 and nco suggesting an across the board deterioration of conditions that effectively will lead to default. It is worth recalling that, at this time horizon, the signal strength of lgr diminishes (see Table 3.3.4.5)

**Figure 5.26.** Failure signals from significant factors in 3 year horizons

![Graph showing odds ratios for exposure classes 3 and 4 over 3 year horizon](image)

**Notes:** On the horizontal axis 3 and 4 stand for the 3rd and 4th risk exposure class respectively

**Figure 5.27.** Failure signals from significant factors in 2 year horizon

![Graph showing odds ratios for exposure classes 3 and 4 over 2 year horizon](image)

**Notes:** On the horizontal axis 3 and 4 stand for the 3rd and 4th risk exposure class respectively
From the graph of Figure 5.28 it is clear the one year prior to failure the set of CAMEL factors conveying a failure signal is the same as during the 2 year horizon. The signals carried over from the exposure class 4 are substantially increased. Note again that, at this time horizon, the signal strength of lgr decreases and in fact dies (see Table 5.20).

**Figure 5.28.** Failure signals from significant factors in 1 year horizon

![1 year horizon: odds ratios for exposure classes 3 and 4](image)

**Notes:** On the horizontal axis 3 and 4 stand for the 3rd and 4th risk exposure class respectively.

Apparently the findings in this Section are verbatim to the ones presented in Section 5.3.2 by exploiting the technique of Correspondence Analysis. We again identify three Stages, one that spans 4-5 years prior to failure, the second spans about a year (3 years prior to failure) and a third one starting about 2 years prior to failure. The CAMEL factors that carry useful information for failure in each Stage are the same as found in Section 5.3.2, that is:

- rbc, lgr and rwa for Stage I
- rbc, lgr, rwa, ncl, roa and cov for Stage II and finally
- rbc, rwa, ncl, roa, cov, equ, nim, eff, past30 and nco for Stage III.

In Figure 5.29 we illustrate the structure of the three Stages in terms of the CAMEL factors that carry useful information for failure in each Stage. Figure 5.30 displays the duration of the three Stages for the life cycle of bank failure.

According to the numerical results presented earlier it is clear that, all risk factors satisfy the *significance* criterion which requires that the risk factor should contain significant forecasting.
power over failure. This is because, as the analysis carried out indicated, there is strong evidence for higher odds of failure for banks belonging to the 3rd and 4th risk exposure classes, where for the 4th risk exposure group the odds of failure increase markedly.

**Figure 5.29.** Structure of the three Stages for the life cycle of bank failure

As far as the *correct direction* criterion is concerned (which requires that the odds of failure are above unity for all risk exposure classes, in comparison to the reference class), it is fulfilled by the vast majority of risk factors, chiefly for risk exposure classes 3 and 4. This is illustrated in Figure 5.31 and Figure 5.32 where the odds ratios of exposure classes 3 and 4 have been plotted for all the CAMEL factors that have been characterized as significant in each of the 3 Stages.

Finally, the *monotonicity* criterion according to which the odds of failure should increase monotonically across the risk exposure classes is also valid for all significant CAMEL factors (in the respective time horizon), with the exception of only one, cov who violated the monotonicity property at the 3 year horizon. Figure 5.33 illustrates that for all the significant risk factors for each time horizon.
**Figure 5.31.** Odds ratios of exposure class 3 for significant CAMEL factors

Notes: On the horizontal axis -20, -16, -12, -8, and -4 denote quarters prior to failure.

**Figure 5.32.** Odds ratios of exposure class 4 for significant CAMEL factors

Notes: On the horizontal axis -20, -16, -12, -8, and -4 denote quarters prior to failure.
**Figure 5.33.** Odds ratios of exposure classes 2, 3, 4 for significant CAMEL factors
A few more things are in order with regards to the behavior of signals across the three stages. The signal strength of lgr diminishes and in fact dies out at STAGE III, while for all other factors, on average, the magnitude and significance of the signals increase as we move closer to failure.
Also, a salient property of signals is that their strength markedly increases for banks belonging to the highest risk exposure category. Table 5.21 depicts the Intensive Margin (mean of the odds of significant factors) and the Extensive Margin (number of significant factors) of each Stage of the life cycle of bank failure. The numbers in parenthesis indicate the Extensive Margin in each of the years included in the respective Stage (year 5 and 4 for Stage I and year 2 and 1 for Stage I).

For a graphical illustration see Figure 5.34. It is evident that a significant increase of the both margins is experienced when shifting from Stage I to Stage III or from Stage II to Stage III. However, in the transition from Stage I to Stage II the value of Intensive Margin remains stationary, a fact that may be attributed to the weak power of the new CAMEL factors that enter in the group in Stage II (as compared to the ones that are already present there from Stage I).

<table>
<thead>
<tr>
<th>Table 5.21. Intensive and extensive margins for each stage</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Stage I</td>
</tr>
<tr>
<td>Stage II</td>
</tr>
<tr>
<td>Stage III</td>
</tr>
</tbody>
</table>

**Figure 5.34** Extensive and Intensive Margins of the three stages of the life cycle of bank failure
5.3.5 Assessing the CAMEL factors forecasting ability by the aid of discriminant analysis

Discriminant Analysis (also known as discriminant function analysis) is a powerful descriptive and classificatory technique that was originally introduced and developed by R. A. Fisher (1936) in order to solve a taxonomic problem using multiple botanical measurements. Later on the techniques of Discriminant Analysis were utilized also for predicting the default risk for banks, loans and securities. The pioneer work in this area could be attributed to Altman (1968), which suggested using the so-called ‘‘Z score’’ to predict firms’ default risk. Since then, his seminal work was followed by hundreds of research articles on this issue (see the review articles: Kumar and Ravi (2007), Fethi and Pasiouras (2009) and Demyanyk and Hasan (2010)).

The problem that is addressed with the discriminant analysis techniques is how well it is possible to separate two or more groups of individuals, given measurements for these individuals on several variables.

In the special case where \( m \) distinct groups are of interest, we should have at hand \( m \) random samples available, one from each group and values on \( p \) characteristics (variables) \( X_1, X_2, \ldots, X_p \) will be available for each sample. The mean vectors for the \( m \) samples are regarded as estimates of the true mean vectors for the groups. Then, according to the most popular discriminant analysis technique, an appropriate distance between an individual and the group centers is calculated and an individual can be allocated to the group that it is closest to him/her. This may or may not be the group that the individual actually came from, so the percentage of correct allocations is clearly an indication of how well groups can be separated using the available variables.

A typical distance exploited in Discriminant Analysis is the so called Mahalanobis distance which makes use of the pooled sample covariance matrix, however other plausible distances may be used as well.

In another setting, Discriminant Analysis could be viewed as a method for classifying (i.e., individuals, subjects, participants) into pre-existing groups based on similarities between that case and the other cases belonging to the groups (this type of Discriminant Analysis is sometimes called predictive discriminant analysis).
As made clear from the above discussion, Discriminant analysis is used in situations where
the clusters are known a priori: the aim of discriminant analysis is to classify an observation, or
several observations, into these known groups. Usually, these groups could have been formed by
a cluster analysis performed on past data.

By way of example let us consider the problem of credit scoring in the banking sector. A
bank knows from past experience that there are good customers (who repay their loan without
any problems) and bad customers (who showed difficulties in repaying their loan). When a new
customer asks for a loan, the bank has to decide whether or not to give the loan. The past records
of the bank provide two data sets: multivariate observations $X_1, X_2, \ldots, X_p$ pertaining to
customer hard and soft info (e.g. age, gender, education, salary, marital status, job type, etc.) on
the two categories of customers. The new customer is a new multivariate observation (case) on
which we have measured the same characteristics (variables). The discrimination rule has to
classify the customer into one of the two existing groups and the discriminant analysis theory
should develop appropriate tools to evaluate the risk of a possible/bad decision.

It is sometimes useful to be able to determine functions of the variables $X_1, X_2, \ldots, X_p$ that
in some sense separate the $m$ groups as well as is possible. The simplest approach involves
taking linear combinations of the original variables. In the special case where only two distinct
groups are of interest ($m=2$), a single linear combination, say

$$ Z = a_1 X_1 + a_2 X_2 + \ldots + a_p X_p $$

(5.32)

is sufficient for our purposes. This is usually referred to as canonical discriminant functions and
is chosen so as it reflects group differences as much as possible. Finding the coefficients of the
canonical discriminant function turns out to be an eigenvalue problem. In the case where we
wish to apply Discriminant Analysis for $m$ groups, we may use up to $m-1$ discriminant functions.

Several tests of significance are useful in conjunction with a Discriminant Analysis. A test
statistic is available to check whether there is a significant difference between the mean values
for any pair of groups, while a likelihood ratio test has been developed to test for overall
differences between the means of the $m$ groups. In addition, a test is available to see whether the
canonical discriminant function $Z$ varies significantly from group to group.
It should be mentioned that, the application of Discriminant Analysis methods, require the assumption that within groups the data follow multivariate normal distributions. However, a failure of our data to meet this assumption does not necessarily mean that Discriminant Analysis is a waste of time. It may well turn out that excellent discrimination is possible on non-normal populations. The only problem is that, in this case, it may not be simple to test statistically the significance of results.


Below we shall exploit the technique of Discriminant Analysis to reconfirm the significance of the groups of CAMEL risk factors identified as significant in predicting bank failure in each stage of the bank lifecycle. The underlying idea in our analysis is that, if a set of factors has a high predicting capability for bank failure at a specific horizon, this set should be also capable to discriminate efficiently the (finally) defaulted and non-defaulted banks. A simple way to verify whether the last assertion holds true is the following: apply Discriminant Analysis on the available data of the CAMEL risk factors, considering the mean vectors for the defaulted and non-defaulted banks as estimates of the true mean vectors for the groups. Then use the discriminant function to classify each bank to either of the two classes (failed/non-failed) and compute the percentage of correct allocations. The last quantities, apparently may be used as an indication of how well groups can be separated using the available variables at the specific time horizon.

Applying the Discriminant Analysis technique at the CAMEL factors that were found significant at Stage I (rbc, rwa and lgr) for a 5 years horizon we get the following formula of the discriminant function (canonical form)

\[ Z = 0.6268612 \cdot rbc - 0.9619586 \cdot roa + 0.0065565 \cdot lgr. \] (5.33)

The group means on \( Z \) for the non-defaulted and defaulted banks was computed as 0.001184 and -0.6769001 respectively. Table 5.22 depicts the probabilities of correct and incorrect classifications by the discriminant function given above. We see that 64.21% of the non-defaulted banks were correctly classified while 35.79% were misclassified. On the other hand, 79.32% of the (finally) defaulted banks were classified in the correct category, based on
the values of the significant CAMEL factors of Stage I; the rest 20.68% were misplaced at the category of non-defaulted banks.

In order to have a basis of comparison for the strength of the CAMEL factors identified as significant failure predictors at Stage I, we proceeded to the application of the Discriminant Analysis technique to the rest CAMEL factors (i.e. those which were found insignificant at Stage I by the analysis carried out in the previous paragraphs) as well as to the whole set of CAMEL factors we have at hand. The probabilities of correct and incorrect classifications by the new discriminant function that arises for the same time period are given in Tables 5.23 and 5.24, respectively.

<table>
<thead>
<tr>
<th>Table 5.22. Probabilities of correct/incorrect classification at a 5 year horizon (significant CAMEL factors)</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>Non-defaulted</td>
</tr>
<tr>
<td>Defaulted</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 5.23. Probabilities of correct/incorrect classification at a 5 year horizon (insignificant CAMEL factors)</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>Non-defaulted</td>
</tr>
<tr>
<td>Defaulted</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 5.24. Probabilities of correct/incorrect classification at a 5 year horizon (all CAMEL factors)</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>Non-defaulted</td>
</tr>
<tr>
<td>Defaulted</td>
</tr>
</tbody>
</table>

A comparison of Tables 5.22-5.24 clearly indicates that, for the time horizon under investigation, roa, rbc and lgr (as a group) perform much better that the rest 8 CAMEL factors in the task of predicting bank failures. It is of interest to note that the group of significant factors of Stage I, seems to do better in failure forecasting, than the whole set of CAMEL factors at
hand. The all-CAMEL factors model developed by the Discriminant Analysis exhibit a slightly better performance in the task of predicting non-defaults.

The foregone discussion offers us a confirmation that the set of CAMEL factors selected at Stage I is appropriate for future failure forecasting at a 5 year time horizon.

Table 5.25 summarizes the results of the application of Discriminant Analysis for all time horizons of interest (5, 4, 3, 2 and 1 year before failure). For each period we have tabulated the probabilities of correct classification when only the significant CAMEL factors of the respective stage are used (shaded entries) as well as the probabilities of correct classification when only the rest CAMEL factors are introduced in the model. We observe that, for all time horizons, the CAMEL factors we have selected for each stage as significant ones, are much more powerful in predicting future failures than the factors that have been left out. Focusing on the first category (significant factors) and looking across the time horizons we observe that, within stages (where the set of significant factors is not altered), there is an improvement in both probabilities when moving closer to failure. There is a small decay in the probability of correct classification for failures at Stage II, however the probability of correct classification for non-failures exhibits a continuous improvement along all the stages and time horizons. As a result the mean probability of correct classification seems to get better when approaching the time of failure, with a slight, rather insignificant, deterioration at Stage II.

| Table 5.25. Probabilities of correct/incorrect classification for each time horizon |
|---------------------------------|-----|-----|-----|-----|-----|
| Stage                           | 5   | 4   | 3   | 2   | 1   |
| Non-defaulted                   | 64.21 | 65.12 | 67.65 | 61.42 | 81.35 | 69.93 | 91.28 | 96.67 |
| Defaulted                       | 79.32 | 61.09 | 79.77 | 63.75 | 57.50 | 39.08 | 61.54 | 77.18 |
| Mean                            | 71.77 | 63.11 | 73.71 | 62.59 | 69.43 | 54.51 | 76.41 | 86.93 |

It is also interesting to note that, the discontinued improvement of the correct classification for non-failures at Stage II, has probably occurred because new CAMEL factors that jumped in at his stage, were not as strong as the ones carried over from Stage I. To make this clear we have calculated in Table 5.26 the probabilities of correct and incorrect classifications at 3 year time horizon (Stage II) when only the significant CAMEL factors of Stage I are used. The probability of correct classification for failures has now arisen to the impressive percentage of 82.22%
which reveals that the 3 significant factors of Stage I, keep carrying significant information for future failures at Stage II as well.

| Table 5.26. Probabilities of correct/incorrect classification at a 3 year horizon (significant CAMEL factors of Stage I) |
|---|---|---|
| True Classified | Non-defaulted | Defaulted |
| Non-defaulted | 67.15 | 32.85 |
| Defaulted | 17.78 | 82.22 |

The general conclusion to be drawn from the analysis of this section is that the groups of CAMEL factors formed with the application of the techniques of Correspondence Analysis and logistic regression in Sections 5.3.2 and 5.3.3 are fully supported by the outcomes of the Discriminant Analysis approach, although the latter does not offer any help in identifying exposure risk classes that provide evidence for future failures.

5.3.6. Prediction of Failure Probabilities per Risk Class

As indicated in Section 3.3.4, the binary variable

\[ D_{i,t} = \begin{cases} 1, & \text{if bank } i \text{ failed at time } t \\ 0, & \text{otherwise,} \end{cases} \]  \hspace{1cm} (5.34)

which describes the status of the \( i \)-th bank (defaulted/non-defaulted) at time \( t \) can be expressed by the logistic regression model as follows

\[ D_{i,t} = \frac{1}{1 + \exp\left[-(a + \beta_2 C_{i,j,t-k}^{(2)} + \beta_3 C_{i,j,t-k}^{(3)} + \beta_4 C_{i,j,t-k}^{(4)}) + u_{i,t}\right]} \]  \hspace{1cm} (5.35)

where \( k = 1,2,3,4,5 \) (years) indicates the horizon checked each time and \( u_{i,t} \) the error term of the model. As a consequence, the default probability of bank \( i \) at time \( t \) \( p_{i,t} = E(D_{i,t}) = \Pr(D_{i,t} = 1) \) can be evaluated by the formula

\[ p_{i,t} = E(D_{i,t}) = \Pr(D_{i,t} = 1) = \frac{1}{1 + \exp\left[-(a + \beta_2 C_{i,j,t-k}^{(2)} + \beta_3 C_{i,j,t-k}^{(3)} + \beta_4 C_{i,j,t-k}^{(4)})\right]} \]  \hspace{1cm} (5.36)

which receives the equivalent form

\[ \logit(p_{i,t}) = \log\left(\frac{p_{i,t}}{1 - p_{i,t}}\right) = a + \beta_2 C_{i,j,t-k}^{(2)} + \beta_3 C_{i,j,t-k}^{(3)} + \beta_4 C_{i,j,t-k}^{(4)}. \]  \hspace{1cm} (5.37)
A further analysis of the last formula, leads to the following result, for the logit transformation of the default probabilities of a bank, in terms of the exposure class it belongs (see also Table 5.15)

\[
\text{logit}(p_{i,t}) = \begin{cases} 
  a & \text{if bank } i \text{ belongs at exposure class } 1, \\
  a + \beta_2 & \text{if bank } i \text{ belongs at exposure class } 2, \\
  a + \beta_3 & \text{if bank } i \text{ belongs at exposure class } 3, \\
  a + \beta_4 & \text{if bank } i \text{ belongs at exposure class } 4.
\end{cases}
\] (5.38)

Making use of the fact that

\[
\text{logit}(p_{i,t}) = \log \left( \frac{p_{i,t}}{1 - p_{i,t}} \right)
\] (5.39)

we may readily arrive at the following expressions for the default probabilities of a bank, in terms of the exposure class it belongs

\[
p_{i,t} = \begin{cases} 
  \exp(-a) & \text{if bank } i \text{ belongs at exposure class } 1, \\
  \exp(-(a + \beta_2)) & \text{if bank } i \text{ belongs at exposure class } 2, \\
  \exp(-(a + \beta_3)) & \text{if bank } i \text{ belongs at exposure class } 3, \\
  \exp(-(a + \beta_4)) & \text{if bank } i \text{ belongs at exposure class } 4.
\end{cases}
\] (5.40)

After having estimated the parameters of the logistic model, we may easily estimate the default probabilities of a bank (per exposure class), by the formula

\[
\hat{p}_{i,t} = \begin{cases} 
  \frac{1}{1 + \exp(-\hat{a})} & \text{if bank } i \text{ belongs at exposure class } 1, \\
  \frac{1}{1 + \exp(-\hat{a} + \hat{\beta}_2)} & \text{if bank } i \text{ belongs at exposure class } 2, \\
  \frac{1}{1 + \exp(-\hat{a} + \hat{\beta}_3)} & \text{if bank } i \text{ belongs at exposure class } 3, \\
  \frac{1}{1 + \exp(-\hat{a} + \hat{\beta}_4)} & \text{if bank } i \text{ belongs at exposure class } 4.
\end{cases}
\] (5.41)

where \(\hat{a}, \hat{\beta}_2, \hat{\beta}_3, \hat{\beta}_4\) are the estimates of parameters \(a, \beta_2, \beta_3, \beta_4\) respectively.

By way of example, let us consider the CAMEL factor rwa for a 5 year (20 quarters) forecasting horizon. The estimates of the parameters \(a, \beta_2, \beta_3, \beta_4\) for the respective logistic model established in Section 3.3.4 are
\[ \hat{a} = -7.839615 \quad \hat{\beta}_2 = 0.0870831, \quad \hat{\beta}_3 = 1.143888, \quad \hat{\beta}_4 = 2.359461; \]  
(5.42)

Note that the estimated values given here for exposure classes 2,3,4 are apparently different from the ones shown in Table 5.16, since the values given there are the odds ratios and not the original coefficients provided by the logistic regression model. Therefore, for a 5 year forecasting horizon, the estimated default probabilities of a bank per exposure class, equal

\[ \hat{p}_{i,t} = \begin{cases} 
\frac{1}{1 + \exp(7.839615)} = 0.00039366 & \text{if bank } i \text{ belongs at exposure class 1,} \\
\frac{1}{1 + \exp(7.839615 - 0.0870831)} = 0.000429469 & \text{if bank } i \text{ belongs at exposure class 2,} \\
\frac{1}{1 + \exp(7.839615 - 1.143888)} = 0.001234657 & \text{if bank } i \text{ belongs at exposure class 3,} \\
\frac{1}{1 + \exp(7.839615 - 2.359461)} = 0.004151382 & \text{if bank } i \text{ belongs at exposure class 4.} 
\end{cases} \]  
(5.43)

Therefore, for a 5 year (20 quarters) forecasting horizon, a bank belonging to exposure class 4 has a predicted failure probability 0.4%, while a bank belonging to exposure class 3 has a predicted failure probability 0.1%. Apparently, for this time horizon, a bank belonging to exposure class 4 has \(0.004151382/0.000393666=10.5\) times higher probability of default as compared to a bank belong to risk class 1, a bank belonging to exposure class 3 has \(0.001234657/0.000393666=3.1\) times higher probability of default, while a bank belonging to exposure class 2 has almost the same failure probability as one that belongs to exposure class 1 \((0.000429469/0.000393666=1.09)\).

In Tables 5.27-5.31 we provide the estimated coefficients of the logistic regression models and the estimated probabilities of bank failure per exposure class for the significant CAMEL risk factor of each stage.

<table>
<thead>
<tr>
<th>Risk factor</th>
<th>Coefficients</th>
<th>Default probabilities ((\times 10000))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\hat{a})</td>
<td>(\hat{\beta}_2)</td>
</tr>
<tr>
<td>Rbc</td>
<td>-7.18248</td>
<td>-0.12684</td>
</tr>
<tr>
<td>RwA</td>
<td>-7.83962</td>
<td>0.08708</td>
</tr>
<tr>
<td>IGR</td>
<td>-7.19182</td>
<td>0.15835</td>
</tr>
</tbody>
</table>

Table 5.27. Coefficients of the logistic regression model and default probabilities per exposure class for 20 quarters forecasting horizon
Table 5.28. Coefficients of the logistic regression model and default probabilities per exposure class for 16 quarters forecasting horizon

<table>
<thead>
<tr>
<th>Risk factor</th>
<th>Coefficients</th>
<th>Default probabilities (×10000)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{a}$</td>
<td>$\hat{\beta}_2$</td>
</tr>
<tr>
<td>Rbc</td>
<td>-7.29408</td>
<td>-0.07066</td>
</tr>
<tr>
<td>Rwa</td>
<td>-8.09378</td>
<td>-0.16255</td>
</tr>
<tr>
<td>lgr</td>
<td>-7.23565</td>
<td>-0.25504</td>
</tr>
</tbody>
</table>

Table 5.29. Coefficients of the logistic regression model and default probabilities per exposure class for 12 quarters forecasting horizon

<table>
<thead>
<tr>
<th>Risk factor</th>
<th>Coefficients</th>
<th>Default probabilities (×10000)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{a}$</td>
<td>$\hat{\beta}_2$</td>
</tr>
<tr>
<td>Rbc</td>
<td>-7.66633</td>
<td>0.22899</td>
</tr>
<tr>
<td>ncl</td>
<td>-6.75983</td>
<td>-0.40586</td>
</tr>
<tr>
<td>roa</td>
<td>-6.88822</td>
<td>-0.28360</td>
</tr>
<tr>
<td>cov</td>
<td>-6.75322</td>
<td>0.03283</td>
</tr>
<tr>
<td>Rwa</td>
<td>-8.74799</td>
<td>0.88264</td>
</tr>
<tr>
<td>lgr</td>
<td>-6.98854</td>
<td>-0.24438</td>
</tr>
</tbody>
</table>

Table 5.30. Coefficients of the logistic regression model and default probabilities per exposure class for 8 quarters forecasting horizon

<table>
<thead>
<tr>
<th>Risk factor</th>
<th>Coefficients</th>
<th>Default probabilities (×10000)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{a}$</td>
<td>$\hat{\beta}_2$</td>
</tr>
<tr>
<td>Rbc</td>
<td>-8.64364</td>
<td>0.81853</td>
</tr>
<tr>
<td>ncl</td>
<td>-7.82956</td>
<td>-0.12520</td>
</tr>
<tr>
<td>equ</td>
<td>-7.39173</td>
<td>0.29260</td>
</tr>
<tr>
<td>roa</td>
<td>-8.31705</td>
<td>0.78357</td>
</tr>
<tr>
<td>cov</td>
<td>-7.56234</td>
<td>0.06250</td>
</tr>
<tr>
<td>nim</td>
<td>-7.14232</td>
<td>-0.26844</td>
</tr>
<tr>
<td>eff</td>
<td>-7.05760</td>
<td>-0.43315</td>
</tr>
<tr>
<td>Past30</td>
<td>-7.39966</td>
<td>-0.02152</td>
</tr>
<tr>
<td>nco</td>
<td>-7.57736</td>
<td>0.23263</td>
</tr>
<tr>
<td>Rwa</td>
<td>-9.28313</td>
<td>0.75389</td>
</tr>
</tbody>
</table>
Table 5.31.  
Coefficients of the logistic regression model and default probabilities per exposure class for 4quarters forecasting horizon

<table>
<thead>
<tr>
<th>Risk factor</th>
<th>Coefficients</th>
<th>Default probabilities (×10000)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \hat{a} )</td>
<td>( \hat{\beta}_2 )</td>
</tr>
<tr>
<td>Rbc</td>
<td>-9.67939</td>
<td>0.51087</td>
</tr>
<tr>
<td>ncl</td>
<td>-9.68358</td>
<td>0.28773</td>
</tr>
<tr>
<td>equ</td>
<td>-9.40238</td>
<td>1.41732</td>
</tr>
<tr>
<td>roa</td>
<td>-8.77374</td>
<td>-0.91640</td>
</tr>
<tr>
<td>cov</td>
<td>-9.33930</td>
<td>0.40548</td>
</tr>
<tr>
<td>nim</td>
<td>-8.22354</td>
<td>0.10924</td>
</tr>
<tr>
<td>eff</td>
<td>-7.56924</td>
<td>-1.07916</td>
</tr>
<tr>
<td>Past30</td>
<td>-8.38250</td>
<td>-0.05132</td>
</tr>
<tr>
<td>nco</td>
<td>-8.59683</td>
<td>-0.25530</td>
</tr>
<tr>
<td>Rwa</td>
<td>-9.40242</td>
<td>1.13960</td>
</tr>
</tbody>
</table>

5.3.7 Principal Components Analysis Models for Risk Factors

Principal Components Analysis (PCA) is amongst the oldest and most widely used techniques of multivariate Analysis. It was originally introduced by Karl Pearson (1857 – 1936) and later on popularized by Harold Hotelling (1895–1973). Pearson (1901) described the method of Principal Components Analysis as a technique for studying specific problems that were of interest to biometricians at that time, although he did not propose a practical method of calculation for more than two or three variables. A development of practical computational methods for applying the method came much later from Hotelling (1933). However, at that time, the calculations were extremely difficult for more than a few variables since they had to be done by hand. It was not until electronic computers became widely available that the technique achieved widespread use.

The scope of the Principal Components Analysis method is to describe the variation of a set of multivariate data in terms of a set of uncorrelated variables each of which is a particular linear combination of the original variables. The lack of correlation is a useful property because it means that the indices measure different 'dimensions' in the data. Technically, we aim at identifying a few linear combinations (termed as components) that account for most of the
variation in the original data. If this is achieved, then they can be used to summarize the data with only a little loss of information, thus providing a reduction in the dimensionality of the data. In some applications, the principal components are often obtained for use as input to another analysis.

The first principal component is the linear combination with maximal variance; we are essentially searching for a direction along which the observations are maximally separated or spread out. The second principal component is the linear combination with maximal variance in a direction orthogonal to the first principal component, and so on. Thus, the new variables defined by the principal components are derived in decreasing order of importance so that, the first principal component accounts for as much as possible of the variation in the original data.

Let us assume, for example, that we wish to rank banks on the basis of their scores on several characteristics, e.g. a number of different CAMEL factors. An average score would provide a single scale on which to compare the banks, however, using unequal weights and possibly contrasting signs we can

a. spread the banks out further on the scale
b. combine characteristics with different monotonicity properties as compared to the event of interest, i.e. default. Some CAMEL factors are expected to indicate improvement of the banks status (with respect to future failure) while others indicate deterioration. Principal Components Analysis can combine them with opposite signs thereof formulating an index looking at the correct direction.

This approach is expected to provide a score (composite index) securing maximum discrimination between the banks, with factors that vary most within the sample of data, being given the highest weight. The final score can then be used to obtain a better ranking and interpreting the behavior of more than one bank characteristics.

When applying Principal Components Analysis there is always the hope that the variances of many of the indices (principal components of higher order) will be so low as to be negligible. In that case the variation in the data set can be adequately described by the first few components with variances that are not negligible. Thus we would economize in terms of
degrees of freedom, since the variation in the original characteristics under study is accounted for by a smaller number of variables.

Apparently, a Principal Components Analysis does not always succeed in reducing a set of a large number of original variables to a small number of transformed variables (components) that can satisfactorily describe the structure of the available data. For example, if the original variables are uncorrelated then the analysis does absolutely nothing. The best results are obtained when the original variables are highly (linearly) correlated. In this case it is quite probable that the original variables can be adequately represented by two or three principal components. If this happens, then the important principal components will be of interest as measures for proceeding to the interpretation of the underlying relations in the data. Moreover, it will also be quite useful to know that there is a good deal of redundancy in the original variables, with most of them measuring similar things.

In Table 5.32 we provide the correlations of the CAMEL factors as obtained from the quarterly data covering the period spanning from the first quarter of 2000 to the fourth quarter of 2011 on the FDIC-insured commercial banks in USA.

<table>
<thead>
<tr>
<th></th>
<th>rbc</th>
<th>ncl</th>
<th>equ</th>
<th>roa</th>
<th>cov</th>
<th>nim</th>
<th>eff</th>
<th>past30</th>
<th>nco</th>
<th>rwa</th>
<th>lgr</th>
</tr>
</thead>
<tbody>
<tr>
<td>rbc</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ncl</td>
<td>-0.049</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>equ</td>
<td>0.733</td>
<td>-0.045</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>roa</td>
<td>0.122</td>
<td>-0.301</td>
<td>0.167</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>cov</td>
<td>0.025</td>
<td>-0.027</td>
<td>0.014</td>
<td>0.089</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>nim</td>
<td>-0.009</td>
<td>-0.108</td>
<td>0.177</td>
<td>0.254</td>
<td>-0.009</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>eff</td>
<td>-0.016</td>
<td>0.116</td>
<td>-0.027</td>
<td>-0.147</td>
<td>-0.012</td>
<td>-0.066</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>past30</td>
<td>-0.003</td>
<td>0.249</td>
<td>0.019</td>
<td>-0.077</td>
<td>-0.019</td>
<td>0.106</td>
<td>0.025</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>nco</td>
<td>-0.014</td>
<td>0.368</td>
<td>0.049</td>
<td>-0.235</td>
<td>-0.022</td>
<td>0.137</td>
<td>0.057</td>
<td>0.131</td>
<td>1.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>rwa</td>
<td>-0.554</td>
<td>0.044</td>
<td>-0.068</td>
<td>0.004</td>
<td>-0.024</td>
<td>0.258</td>
<td>-0.011</td>
<td>0.053</td>
<td>0.088</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>lgr</td>
<td>0.001</td>
<td>-0.002</td>
<td>0.003</td>
<td>0.001</td>
<td>0.000</td>
<td>0.005</td>
<td>0.003</td>
<td>-0.002</td>
<td>0.000</td>
<td>-0.001</td>
<td>1.000</td>
</tr>
</tbody>
</table>

It is clear that some of the CAMEL factors are highly correlated (e.g. equ and rbc with respective correlation coefficient 73.3%) while others exhibit medium (e.g. rbc and rwa with correlation coefficient -55.4%) or very low linear correlation (e.g. lgr and rwa with correlation coefficient -0.1%).
In view of these observations we may state that Principal Components Analysis can offer useful insight when linear combinations of CAMEL factor are sought in order to form indices with augmented predicting power on future bank default.

Let us next proceed to a brief description of the calculations involved in a Principal Components Analysis and then use this technique to our CAMEL factors data. For more details on the mathematics of the Principal Components Analysis technique, the interested reader is referred to Manly (1986), Everitt and Dunn (1991) or Johnson and Wichern (1998).

A Principal Components Analysis starts with data on \( p \geq 2 \) variables \( X_1, X_2, \ldots, X_p \) for \( n \) individuals, as indicated in the next matrix

\[
X = \begin{bmatrix}
X_{11} & \cdots & X_{1f} & \cdots & X_{1p} \\
\vdots & \ddots & \vdots & & \vdots \\
X_{ni} & \cdots & X_{nj} & \cdots & X_{np} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
X_{ni} & \cdots & X_{nj} & \cdots & X_{np}
\end{bmatrix}_{n \times p}
\]

The first principal component is a linear combination of the variables

\[
Y_1 = a_1 X_1 + a_2 X_2 + \ldots + a_p X_p
\]

that has the highest possible variance (as estimated from the available data), subject to the condition that

\[
\|a\| = \sum_{i=1}^{p} a_i^2 = 1
\]

where \( a = (a_1, a_2, \ldots, a_p) \). The constraint on the coefficients vector \( a \) is introduced because if this is not done then the variance of \( Y \) can be unboundedly increased by simply multiplying all \( a_1, a_2, \ldots, a_p \) by the same large number. The second principal component, is a second linear combination of the variables

\[
Y_2 = b_1 X_1 + b_2 X_2 + \ldots + b_p X_p
\]

such that its variance is as large as possible subject to the constraint that
\[
\sum_{i=1}^{p} b_i^2 = 1; \quad (5.48)
\]

moreover, \( Y_1 = a_1 X_1 + a_2 X_2 + \ldots + a_p X_p \) and \( Y_2 = b_1 X_1 + b_2 X_2 + \ldots + b_p X_p \) should be uncorrelated, a condition that is met if the following restriction holds true

\[
a_1 b_1 + a_2 b_2 + \ldots + a_p b_p = 0. \quad (5.49)
\]

Further principal components can be introduced by continuing in the same way. If we start with \( p \geq 2 \) variables then one can build up to \( p \) principal components.

Principal Components Analysis involves finding the eigenvalues and eigenvectors of the sample covariance matrix \( C \). We recall that the diagonal elements of this matrix contain the sample variances of the \( p \) original variables \( X_1, X_2, \ldots, X_p \) and the off-diagonal elements of it contain the sample covariances of them. This matrix can be expressed as \( Z^T Z / (n-1) \) where \( Z \) is the matrix containing the “centralized” version of the available data, namely

\[
Z = \begin{bmatrix}
    x_{11} - \bar{x}_1 & x_{12} - \bar{x}_2 & \cdots & x_{1p} - \bar{x}_p \\
    x_{21} - \bar{x}_1 & x_{22} - \bar{x}_2 & \cdots & x_{2p} - \bar{x}_p \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{n1} - \bar{x}_1 & x_{n2} - \bar{x}_2 & \cdots & x_{np} - \bar{x}_p 
\end{bmatrix} \quad (5.50)
\]

The sample means appearing above are given by

\[
\bar{x}_j = \frac{1}{n} \sum_{i=1}^{n} x_{ij}, \quad j = 1, 2, \ldots, p. \quad (5.51)
\]

The Principal Components Analysis method calls for the computation of the eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_p \) and the corresponding eigenvectors \( u_1, u_2, \ldots, u_p \) of the sample covariance matrix \( C \). Note that, matrix \( C \) has \( p \) non-negative eigenvalues of these, some of which may be zero (negative eigenvalues are not possible for a covariance matrix). Assuming that the eigenvalues are ordered in decreasing order as follows \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \geq 0 \), then \( \lambda_i \) corresponds to the \( i \)-th principal component, and the coefficients of linear combination describing the \( i \)-th principal component are given by the corresponding eigenvector \( u_i \), \( i = 1, 2, \ldots, p \). The variances of the principal components are simply the eigenvalues of the matrix \( C \).
An important property of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$ is that they add up to the sum of the diagonal elements (the trace) of $C$. This means that the sum of the variances of the principal components is equal to the sum of the variances of the original variables. Therefore, in a sense, the whole set of the $p$ principal components account for all of the variation in the original data. Moreover, the eigenvalue for a principal component indicates the variance that it accounts for out of the total variances. Hence the ratio

$$\frac{\lambda_1}{\lambda_1 + \lambda_2 + \ldots + \lambda_p}$$

(5.52)

offers a measure of the proportion of the total variance that the first principal component accounts for,

$$\frac{\lambda_2}{\lambda_1 + \lambda_2 + \ldots + \lambda_p}$$

(5.53)

indicates the proportion of the total variance that the second principal component accounts for, and so on. The cumulative proportion of the total variance that the first and second principal components account for, equals to

$$\frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + \ldots + \lambda_p}$$

(5.54)

Analogous expressions can be stated for the remaining principal components and the combinations of them.

In order to avoid one variable having an excessive influence on the principal components it is usual to code the variables $X_1, X_2, \ldots, X_p$ so that they have unit variances (this is a typical standardization procedure). The matrix $C$ then takes the form of a correlation matrix, that is, its diagonal elements will be 1’s and the off-diagonal elements of it will now contain the sample correlations of the $p$ original variables $X_1, X_2, \ldots, X_p$. In that case, the sum of the diagonal terms, and hence the sum of the eigenvalues, is equal to $p$, the number of variables.

Summarizing the foregone analysis, we may state the steps in a Principal Components Analysis as follow:

- Start by coding (if needed) the variables $X_1, X_2, \ldots, X_p$ to have unit variances.
➢ Calculate the covariance matrix \( C = ZZ/(n-1) \) (if variable standardization is not necessary) or the correlation matrix (if variable standardization is needed).

➢ Find the eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \geq 0 \) of \( C \) and the corresponding eigenvectors \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \).

➢ The coefficients of the \( i \)-th principal component are then given by \( \mathbf{u}_i \), \( i = 1, 2, \ldots, p \) while \( \lambda_i \) is its variance.

➢ Keep as many components as necessary to have a large proportion of the original variance reflected in them. Discard any components that account only for a small proportion of the variation in the data.

We shall now use the Principal Components Analysis technique to create the linear combination of the CAMEL factors that we identified as significant (in predicting bank failure over a specific period) for each of the three Stages defined in Section 5.3.4. A successful application of the Principal Components Analysis would lead to a small number of transformed variables (components) that can satisfactorily describe the structure of available data. If the first principal component is powerful enough (in the sense that the percentage of the variance it accounts for is high) we could use it as a single index for assessing bank failure over the horizon we are interested in. If not, we can couple it with the second principal component, thereof obtaining a pair of indices that could replace the original set of CAMEL factors that were identified as significant. Apparently this approach is of major interest for Stages II and II where the set of significant CAMEL factors is large.

Let us start by applying the Principal Components Analysis technique at the 3 significant CAMEL factors identified in Stage I, i.e. rbc, rwa and lgr. In view of Table 5.32, at least 2 of them, namely rbc and rwa, have a non-negligible correlation (the sample correlation coefficient is -0.554), therefore we have good material for exploiting the Principal Components Analysis.

From the output of the Principal Components Analysis given in Table 5.33 we see, that the first principal component explains a variance of 1.44147 whereas the other two components absorb variances very much less than this (0.999995 and 0.558539).

[Table 5.33 in Appendix]
Thus the first principal component accounts for $1.44147/(1.44147+0.999995+0.558539)$
100% = 48.05% of the total variance appearing in our dataset, the second for 33.33% and the
third only for 18.62. Clearly the first component is far more important than the other two.

The corresponding eigenvectors are shown in Table 5.34. These eigenvectors provide the
coefficients of the principal components, therefore, the first component is calculated to be

\[ PC_1 = 0.7071 \cdot rbc - 0.7071 \cdot rwa + 0.0095 \cdot lgr. \]  

(5.55)

[Table 5.34 in Appendix]

It is noteworthy that $PC_1$ attempts to combine characteristics with different monotonicity
properties as compared to bank default; an increase in rbc apparently indicates improvement of
the banks status (with respect to future failure) while rwa indicates deteriorations. Principal
Components Analysis has attributed opposite signs in those two factors thereof formulating a
plausible index that may provide overall information on a bank’s status. We should also mention
that the scoring coefficient of lgr is by far smaller than the coefficients of the other two factors.
The situation is reversed in the second principal component where lgr gets the biggest
coefficient while the coefficients of the other two factors are much smaller.

The importance and the interpretation of the score provided by the principal components
can be assessed by looking at the correlation coefficient between the original variables (rbc, rwa,
lgr) and the “fictitious” variables generated by the Principal Components Analysis. This
information is provided in Table 5.35 where one can see that the correlation between rbc and the
first principal component is large and positive (84.90%), the correlation between rwa and the
first principal component is large and negative (-84.89%), while the correlation between lgr and
the first principal component is very small (1.14%). Note also that the correlation between rbc,
raw and and the first principal component is very significant ($p$-value<1%).

<table>
<thead>
<tr>
<th>CAMEL factor</th>
<th>First principal component</th>
<th>Second principal component</th>
<th>Third principal component</th>
</tr>
</thead>
<tbody>
<tr>
<td>rbc</td>
<td>0.849***</td>
<td>-0.000480</td>
<td>0.528***</td>
</tr>
<tr>
<td>rwa</td>
<td>-0.849***</td>
<td>0.0129***</td>
<td>0.528***</td>
</tr>
<tr>
<td>lgr</td>
<td>0.0114</td>
<td>1.000***</td>
<td>-0.00656*</td>
</tr>
</tbody>
</table>
We can now use a reduced set of indices (principal component 1 or both principal components 1 and 2) as a representative index which can replace the 3 original variables that have been identified as significant for failure bank prediction at Stage I.

Our next step is to repeat the analysis exercised in the Sections 5.3.1 and 5.3.3 using the first principal component, instead of each one of the 3 CAMEL factors. To be more specific, we are going to form 4 risk exposure classes for $PC_1$, using the quartiles of the transformed data and then

a. develop a logistic model for default (response) versus the $PC_1$-exposure class for lagged transformed data of a bank.

b. get a crosstabulation of defaulted/non-defaulted banks versus $PC_1$ risk exposure classes.

Thus, we can check the efficacy of the $PC_1$ score in providing an index with forecasting ability. The analysis we are carrying out here, will reveal whether $PC_1$ is capable of providing signals for groups of banks exposed to certain level of the “combined risk factor” defined by it.

The quartiles of the transformed data for stage I (according to (5.12)) are computed as $-0.7399647$, $-0.2245$ and $0.4695836$, so the 4 exposure classes are defined as follows

<table>
<thead>
<tr>
<th>Exposure Class</th>
<th>$PC_1$ Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class 1</td>
<td>$PC_1 \leq -0.7399647$</td>
</tr>
<tr>
<td>Class 2</td>
<td>$-0.7399647 &lt; PC_1 \leq -0.2245$</td>
</tr>
<tr>
<td>Class 3</td>
<td>$-0.2245 &lt; PC_1 \leq 0.4695836$</td>
</tr>
<tr>
<td>Class 4</td>
<td>$PC_1 &gt; 0.4695836$.</td>
</tr>
</tbody>
</table>

Next we generate a dummy variable describing a bank's exposure to each particular risk class at time $t$, as follows

$$C_{i,t} = \begin{cases} 1, & \text{if bank } i \text{ belongs at time } t \text{ to the } r-th \text{ risk exposure class of } PC_1 \\ 0, & \text{otherwise} \end{cases} \quad (5.56)$$

for $r = 1,2,3,4$. A logit model is then established using as explanatory variables the three risk exposure classes, having the lowest risk exposure class as a reference group. The dependent variable is the indicator (dummy) variable.
\[ D_{i,t} = \begin{cases} 1, & \text{if bank } i \text{ failed at time } t \\ 0, & \text{otherwise,} \end{cases} \] (5.57)

which describes the status of the \( i \)-th bank (defaulted/non-defaulted) at time \( t \). The logistic regression model is as follows.

\[
D_{i,t} = \frac{1}{1 + \exp\left[-\left(a + \beta_1 C_{i,t-k}^{(2)} + \beta_2 C_{i,t-k}^{(3)} + \beta_3 C_{i,t-k}^{(4)}\right) + u_{i,t}\right]} 
\] (5.58)

where \( k = 1,2,3,4,5 \) (years) indicates the horizon checked each time and \( u_{i,t} \) the error term of the model.

The row referring to year 5 in Table 5.36 reports the estimation results from the logit model across five years prior to failure.

<table>
<thead>
<tr>
<th>Horizon (in years)</th>
<th>Exposure</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>( R^2 )</th>
<th>Wald</th>
<th>Log likelihood</th>
<th>obs</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
<td>1.9979</td>
<td>4.0767***</td>
<td>15.8786***</td>
<td>0.0675</td>
<td>321.24***</td>
<td>-2220.6286</td>
<td>185564</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>1.6886</td>
<td>2.9250**</td>
<td>13.8650***</td>
<td>0.0680</td>
<td>350.89***</td>
<td>-221536</td>
<td>221536</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>3.9208***</td>
<td>7.9517***</td>
<td>38.9072***</td>
<td>0.0871</td>
<td>364.52***</td>
<td>-1910.6706</td>
<td>181737</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>2.1396</td>
<td>3.8781***</td>
<td>30.3342***</td>
<td>0.0977</td>
<td>459.21***</td>
<td>-2119.9258</td>
<td>214116</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>0.1433</td>
<td>1.1261</td>
<td>49.2683</td>
<td>0.1558</td>
<td>834.77***</td>
<td>-2261.1089</td>
<td>247184</td>
</tr>
</tbody>
</table>

Notice first that the explanatory power of \( PC_1 \) is 6.75% a figure which is higher than the explanatory power of any of the three individual factors that have been combined in the first principal axis; we recall that, at this time horizon, the explanatory power of the significant factors were computed as 2.77% (rbc), 3.2% (lg) and 5.9% (rwa).

Next, observe that, for banks belonging to the 3rd or 4th risk category of \( PC_1 \) the odds of failure are 4.08 and 15.87 times higher compared to the 1st risk category. These figures are much higher than the ones obtained for rbc, rwa and lgr separately (for banks belonging to the 3rd or 4th risk category of rbc the odds of failure were 1.93 and 4.27 times higher as compared to the 1st risk category, for the 3rd and 4th risk category of rwa, the odds of failure were 3.14 and 10.59 times higher, and finally for the 3rd and 4th risk category of lg, the odds of failure were 1.91 and 5.22 times higher).

Looking at the estimation results for the 4-year horizon (see row referring to year 4 in Table 5.35) we find a similar picture to that for the 5-year. Essentially the odds are relatively
close to those for the 5-year horizon, while the explanatory power of $PC_1$ has been improved, but not that much. For banks belonging to the 3rd or 4th risk category of $PC_1$ the odds of failure are 2.92 and 13.87 times higher compared to the 1st risk category. The statistics of $PC_1$ for this time horizon, are better than the respective statistics of rbc and lgr, and comparable to the statistics of rwa (for the same period).

Let us now apply the Principal Components Analysis technique at the 6 significant CAMEL factors that are identified in Stage II, i.e. rbc, ncl, equ, cov, rwa and lgr. As illustrated in Table 5.31, many pairs of them have non-negligible correlations; therefore Principal Components Analysis should be expected to produce strong components that can facilitate the analysis of bank failure signals.

From the output of the Principal Components Analysis given in Table 5.37 we see, that the first principal component explains a variance of 1.4755, the second 1.25373 and the third 1.00006.

[Table 5.37 in Appendix]

The first principal component accounts for 24.59% of the total variance appearing in our dataset, the second for 20.9% and the third 20.90%.

The corresponding eigenvectors are shown in Table 5.38, standardized so that the sum of the squares of the coefficients is unity for each one of them.

[Table 5.38 in Appendix]

These eigenvectors provide the (scoring) coefficients of the principal components, therefore, the first component is calculated to be

$$PC_1 = 0.6075rbc - 0.3757ncl + 0.4204roa + 0.1525cov - 0.5383rwa + 0.0047lgr.$$  (5.59)

One may write similar expressions for the rest of the principal components. It is noteworthy that $PC_1$ combines six characteristics with different monotonicity properties and has succeeded in attributing the correct signs to all of them thereof formulating a plausible index that may provide overall information on a bank’s status with respect to default. The scoring
coefficient of lgr still remains substantially smaller than the coefficients of the other factors. The factor lgr attains high importance only at the third principal component.

The importance and the interpretation of the scores provided by the principal components can be assessed by looking at the correlation coefficient between the original variables (rbc, ncl, equ, cov, rwa and lgr) and the “synthetic” variables generated by the Principal Components Analysis. This information is provided in Table 5.39 where the correlation concerning the first 3 components is given. One can see that the first principal component is highly positively correlated to rbc (73.80%), roa (51.06%) while it exhibits high negative correlation with ncl (-45.64%) and rwa (-65.39%). Therefore the scores observed at this axis will be chiefly affected by the aforementioned four factors. Note also that the correlation between all the significant factors identified in Stage II and the first principal component is very significant ($p$-value$<1\%$), with the exception of lgr ($1\%<p$-value$<5\%$).

<table>
<thead>
<tr>
<th>CAMEL factor</th>
<th>First principal component</th>
<th>Second principal component</th>
<th>Third principal component</th>
</tr>
</thead>
<tbody>
<tr>
<td>rbc</td>
<td>0.738***</td>
<td>-0.394***</td>
<td>0.000881</td>
</tr>
<tr>
<td>ncl</td>
<td>-0.456***</td>
<td>-0.618***</td>
<td>-0.0149***</td>
</tr>
<tr>
<td>roa</td>
<td>0.511***</td>
<td>0.624***</td>
<td>-0.000790</td>
</tr>
<tr>
<td>cov</td>
<td>0.185***</td>
<td>0.208***</td>
<td>-0.0684***</td>
</tr>
<tr>
<td>rwa</td>
<td>-0.654***</td>
<td>0.533***</td>
<td>0.000104</td>
</tr>
<tr>
<td>lgr</td>
<td>0.00571**</td>
<td>0.00583**</td>
<td>0.998***</td>
</tr>
</tbody>
</table>

The row referring to year 3 in Table 5.36 reports the estimation results for the logit model based on the $PC_1$ of Stage II (three years prior to failure). The explanatory power of $PC_1$ is 8.71% a figure which is much higher than the explanatory power of any of the six individual factors that have been combined in the first principal axis. At this time horizon, the explanatory power of the significant factors were computed as 3.23% (rbc), 0.9% (ncl), 1.42% (roa), cov (0.78%), 7.8% (rwa) and 1.97% (lg); apparently only rwa owns an explanatory power competitive to the power of $PC_1$.

Next, observe that, for banks belonging to the 3rd or 4th risk category of $PC_1$ the odds of failure are 7.95 and 38.9 times higher compared to the 1st risk category. These figures are much higher than the ones obtained for rbc, ncl, roa, cov, rwa and lgr separately (for banks belonging
to the 3rd or 4th risk category of rbc the odds of failure were 2.41 and 5.9 times higher as compared to the 1st risk category, for the 3rd and 4th risk category of ncl, the odds of failure were 0.8 and 1.77 times higher, for the 3rd and 4th risk category of roa the odds of failure were 0.8 and 2.2 times higher, for the 3rd and 4th risk category of rwa, the odds of failure were 4.1 and 23.6 times higher, and finally for the 3rd and 4th risk category of lg, the odds of failure were 1.2 and 3 times higher). Apparently the signal conveyed from $PC_1$ about an upcoming failure in the next 3 years is substantially higher than the signal given by any of the individual significant CAMEL factors of this stage.

Let us finally apply the Principal Components Analysis technique at the significant CAMEL factors identified in Stage III, i.e. for all factors apart from lgr. Table 5.40 provides the percentages of total variance explained by each component as well as the cumulative percent as we include additional components in the modeling.

[Table 5.40 in Appendix]

The first principal component accounts for 20.41% of the variance appearing in our dataset, the second for 16.91% and the third 14.63%. Therefore, a combined use of three principal components accounts for more than half of the total variance, which in fact is not bad, given that now we are trying to reduce the dimensionality from 10 to 3.

The corresponding (scoring) coefficients are shown in Table 5.41.

[Table 5.41 in Appendix]

Notice again that $PC_1$ has succeeded in attributing the correct signs to 10 characteristics combines with different monotonicity properties thereof formulating a very powerful index that may provide overall information on a bank’s status.

The importance and the interpretation of the scores provided by the principal components can again be assessed by inspecting the correlation coefficient between the 10 original variables and the “synthetic” variables generated by the Principal Components Analysis which is provided in Table 5.42 (for the first 3 components only). One can see that the first principal component is highly positively correlated to rbc (86.32%), equ (72.53%) and roa (43.96%), while it exhibits negative correlation with ncl (-38.72%), eff (-15.16%), past30 (-17.32%), nco (-27.87%) and
rwa (-52.82%). Therefore the scores observed at this axis will be primarily affected by the aforementioned 8 factors. In addition, the correlation between all the significant factors identified in Stage III and the first principal component is very significant ($p$-value<1%).

The rows referring to years 2 and 1 in Table 5.36 report the estimation results for the logit model based on the $PC_1$ of Stage III (two and one year prior to failure respectively).

The explanatory power of $PC_1$ is 9.77% for 2 years horizon and increases to 15.58% one year before failure. What is impressive, but not surprising (given that the failure is extremely close), is the volume of the signal sent out by risk exposure class 4 of $PC_1$. Banks belonging to the 4th risk category of $PC_1$ have odds of failure 49.27 times higher than their counterparts in 1st risk category! Such a signal is not conveyed by any of the individual CAMEL factors.

For banks belonging to the 3rd or 4th risk category of $PC_1$ two years before failure, the odds of failure are 3.87 and 30.33 times higher compared to the 1st risk category. They are quite high but not as high as the respective odds at the 3 years horizon. This might be attributed to the fact that too-many factors have been incorporated in $PC_1$, some of which are not very strong, and as a consequence, the power of the first principal component at this time horizon is relaxed. However, when the failure comes closer (1 year horizon) the signals of all individual CAMEL factors strengthen and the overall power of $PC_1$ explodes, at least for the risk exposure class where the odds of failure raises to 49.27.

<table>
<thead>
<tr>
<th>CAMEL factor</th>
<th>First principal component</th>
<th>Second principal component</th>
<th>Third principal component</th>
</tr>
</thead>
<tbody>
<tr>
<td>rbc</td>
<td>0.863***</td>
<td>0.4381***</td>
<td>-0.00614***</td>
</tr>
<tr>
<td>nel</td>
<td>-0.3872***</td>
<td>0.6661***</td>
<td>0.0922***</td>
</tr>
<tr>
<td>equ</td>
<td>0.7253***</td>
<td>0.3205***</td>
<td>0.371***</td>
</tr>
<tr>
<td>roa</td>
<td>0.4396***</td>
<td>-0.5215***</td>
<td>0.301***</td>
</tr>
<tr>
<td>cov</td>
<td>0.0998***</td>
<td>-0.0912***</td>
<td>-0.0101***</td>
</tr>
<tr>
<td>nim</td>
<td>0.0844***</td>
<td>-0.1962***</td>
<td>0.801***</td>
</tr>
<tr>
<td>eff</td>
<td>-0.1516***</td>
<td>0.2918***</td>
<td>-0.154***</td>
</tr>
<tr>
<td>past30</td>
<td>-0.1732***</td>
<td>0.3675***</td>
<td>0.361***</td>
</tr>
<tr>
<td>nco</td>
<td>-0.2787***</td>
<td>0.5606***</td>
<td>0.384***</td>
</tr>
<tr>
<td>rwa</td>
<td>-0.5282***</td>
<td>-0.3150***</td>
<td>0.532***</td>
</tr>
</tbody>
</table>
In closing this Section we present Table 5.42 which is similar to Table 5.3. Table 5.43 can be used to assess the strength of $PC_1$ as predictor of bank failures for a specific time horizon (5, 4, 3, 2 and 1 years). The structure of the table is similar to Table 3.3.1.2, the only difference being that, instead of tabulating the results of each individual factor, we provide the failures per exposure class defined by $PC_1$ for all horizons of interest.

<table>
<thead>
<tr>
<th>Horizon Exp. class</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17</td>
<td>20</td>
<td>7</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>32</td>
<td>32</td>
<td>27</td>
<td>19</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>61</td>
<td>53</td>
<td>53</td>
<td>35</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>214</td>
<td>241</td>
<td>193</td>
<td>249</td>
<td>339</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>324</td>
<td>346</td>
<td>280</td>
<td>312</td>
<td>355</td>
</tr>
<tr>
<td><strong>Chi-square</strong></td>
<td>381.5***</td>
<td>420.7***</td>
<td>450.9***</td>
<td>556.8***</td>
<td>957.9***</td>
</tr>
</tbody>
</table>

By reading across each exposure level, one can see how many banks failed for each time horizon. For instance, out of the 324 banks that have failed in a 5 years horizon, only 17 belonged to the first exposure level of $PC_1$, 32 belonged to the second exposure level, 61 belonged to the third exposure level while 2/3 of them (214) belonged to the fourth exposure level. From these figures it is intuitively clear that the explanatory power of $PC_1$, as predictor of failure in a 5 years’ time horizon, is quite large. Similar conclusions hold true for the other 4 time horizons, with the last one (1 year before failure) making the first principal component extremely powerful (339 out of 355 banks that failed belonged to the fourth exposure level of $PC_1$)

Below the Total row, we have provided the Chi-Square statistic as calculated when analyzing the contingency table arising from the failed/non-failed banks distribution across the 4 exposure classes. The higher the Chi-Square statistic, the better for $PC_1$ as a predictor of bank failures. In all time horizons studied the Chi-square statistic is statistically very significant ($p$-value<1%) and therefore, the first principal component seem to a very power predictors for a failure in all time horizons.

Table 5.44 depicts that same information as Table 5.42 in terms of percentages, instead of pure numbers of failures. These results are further illustrated in Figure 5.35
Table 5.44. Comparison of failure proportions (%) per exposure class of $PC_1$ for all horizons

<table>
<thead>
<tr>
<th>Horizon</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.25</td>
<td>5.78</td>
<td>2.50</td>
<td>2.88</td>
<td>1.97</td>
</tr>
<tr>
<td>2</td>
<td>9.88</td>
<td>9.25</td>
<td>9.64</td>
<td>6.09</td>
<td>0.28</td>
</tr>
<tr>
<td>3</td>
<td>18.83</td>
<td>15.32</td>
<td>18.93</td>
<td>11.22</td>
<td>2.25</td>
</tr>
<tr>
<td>4</td>
<td>66.05</td>
<td>69.65</td>
<td>68.93</td>
<td>79.81</td>
<td>95.49</td>
</tr>
<tr>
<td>Total</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
</tr>
</tbody>
</table>

Chi-square: 381.5***, 420.7***, 450.9***, 556.8***, 957.9***

Figure 5.35. Failure proportions (%) per exposure class of $PC_1$ for all horizons

Notes: On the horizontal axis -20, -16, -12, -8, and -4 denote quarters prior to failure.

Based on the above findings we may conclude that combining at each Stage, the significant CAMEL factors that contain significant ex ante explanatory power over future bank failures, we can create a much more powerful single index and define new risk exposure levels by the aid of which particular groups of banks (those belonging to the worst exposure levels) can be identified as being ex ante more susceptible to failure.
5.4. CONCLUSIONS

In this chapter we tested empirically the long term forecasting ability for bank failures of a number of CAMEL risk factors. Our aim was to investigate how many quarters prior to failure they may contain useful signals for the upcoming failure and elucidate whether distinct periods can be identified based on the groups of factors producing failure signals. The novelty of our analysis is that it focuses on signals pertaining not to specific banks, but rather to groups of banks exposed to certain level of the risk factors that are expected to drive failure. Our empirical findings are particularly useful both for market participants, but especially for the regulator.

The dataset that was used is a panel dataset whose cross section consists of all US commercial banks and its time series dimension of the period 2000:q1-2011:q4 (quarter intervals). A preliminary analysis of the available data, revealed that, for all the CAMEL metrics considered, there are no substantial differences between subsequently failed banks and their non-failed peers up until about 20 quarters prior to failure. Therefore, our interest shifted to the period spanning one to five years before failure.

In order to proceed to our empirical results we constructed four exposure risk classes by the aid of the quartiles of each risk factor. More specifically, the cut-points (quartiles) for the risk exposure classes of each risk factor were calculated for time horizons 5, 4, 3, 2 and 1 year before the reference point and each bank was classified in the appropriate class, using the relevant information provided by the associated risk factor.

A preliminary analysis, based on the distribution of future failures in the 4 risk exposure classes, revealed that there are CAMEL factors that contain significant ex ante explanatory power over future bank failures. However the strength of the factors is different in the 5 time horizons examined and the based on that one may probably distinguish different stages based on the groups of CAMEL risk factors from which failure signals are received.

Our next step was to use the Correspondence Analysis technique to analyze the connection between bank failure and the categorical variables describing the four exposure levels of the CAMEL risk factors under a multivariate framework. This approach has the potential to identify associations between categorical variables and provide useful information on their effect on supplementary variables, such as bank failure/non-failure. The findings of the Correspondence
Analysis outcomes reveal that, according to the group of CAMEL factors that send out signals for bank failure, we may think of breaking up the bank life cycle (prior to failure) to the following 3 stages: Stage I which spans the period 4-5 years prior to failure where rbc, lg and rwa, carry useful information for failure; Stage II, 3 years prior to failure, where a new set of significant predictors kick in (ncl, roa, cov and probably nco). The strength of the factors identified in the first Stage, seem to keep carrying useful information on bank failure, for a 3 year horizon as well, however this is not fully substantiated and a further analysis is required. Finally, Stage III starting about 2 years prior to failure when signals are received from all CAMEL factors except lgr, which, at this stage does not seem carrying useful information about the approaching failure.

Since the Correspondence Analysis technique does not provide any quantification of the significance of the factors that have been identified as useful in each Stage, we proceeded at a further analysis pertaining to the development of logistic models. A different logit model, containing as explanatory variables the three higher risk exposure classes (having the lowest risk exposure class as a reference group), was established for each risk factor. The model was applied to five different horizons prior to failure: 5 years (20 quarters), 4 years (16 quarters), 3 years (12 quarters), 2 years (8 quarters), and 1 year (4 quarters).

The fitted model was then exploited to identify which risk exposure class is more (less) vulnerable to failure for each time horizon, as reflected in the failure rates between risk exposure classes. The assessment of the comparison of each exposure class to risk class 1 (baseline) will be carried out by observing the odds ratios after the estimation procedure is completed for each time horizon.

The empirical results provided by the logit models confirmed the findings of the Correspondence Analysis, clarifying the question posed on the strength at Stage II, of the factors identified in Stage I and quantify the forecasting capability of each factor at each time horizon.

We again identified three Stages, one that spans 4-5 years prior to failure, the second includes only 1 year (3 years prior to failure) and a third one starting about 2 years prior to failure. The CAMEL factors that were found to carry useful information for failure in each Stage are as follows:
• rbc, lgr and rwa for Stage I
• rbc, lgr, rwa, ncl, roa and cov for Stage II, and finally
• rbc, rwa, ncl, roa, cov, equ, nim, eff, past30 and nco for Stage III.

Our next task was to compute estimated probabilities of bank failure per exposure class for the significant CAMEL risk factor of each Stage. A further validation, in a multivariate framework, of the importance of the groups of CAMEL risk factors considered as significant in each stage, was established by practicing the Discriminant Analysis method.

Finally, in order to create a smaller number of indices that combine the CAMEL factors that were identified as significant in predicting bank failure over a specific period, we applied Principal Components Analysis and developed logit models using the quartiles of the linearly transformed data. The statistical analysis of the results revealed that, upon combining the significant CAMEL factors that contain significant ex ante explanatory power over future bank failures, we can create much more powerful indices. We may then define new risk exposure levels by the aid of which particular groups of banks (those belonging to the worst exposure levels) can be identified as being ex ante more susceptible to failure. Therefore, based for example in the first principal component which is by far the most powerful one, and the risk exposure classes associated with it, we can have a single index for failure forecasting in each of the three stages of bank failure lifecycle.

Summarizing the material presented in the present chapter, we can say that all our empirical results fully support the suggestion of breaking up the bank life cycle prior to failure to three stages: 4-5 years prior to failure, 3 years prior to failure and finally 1-2 years prior to failure. In the first stage, most of the risk factors (8 out of 11) do not contain any significant forecasting power, while the rest 3 (rbc, lgr and rwa) are quite informative and define a risk class which is very vulnerable to failure. Looking at a 3 year horizon before failure we uncover that the set of significant predictors is augmented by three more factors (ncl, roa and cov). Finally, the empirical results across CAMEL metrics two years prior to failure indicate that most of the risk factors (10 out of 11) contain significant forecasting power. Needless to say, the conclusions drawn in this chapter may be proved quite helpful for the supervisory monitoring of bank conditions.
PART III. RISK MODELING
CHAPTER 6. A POLYNOMIAL LOGISTIC DISTRIBUTION
AND ITS APPLICATIONS IN FINANCE

6.1. INTRODUCTION

The importance of Logistic distribution has been widely recognized in many applied areas such as, Demography, Population Studies, Finance, Agriculture etc. Since its introduction as a model, much attention has been paid to the study of several generalizations of it, which would offer additional flexibility when data fitting is chased. In the present chapter we introduce and develop a natural generalization of the Logistic distribution by considering a probability model whose logit cumulative distribution function transformation is of polynomial type. The performance of the model’s fitting to financial data, using different parameter estimation methods, is also investigated.

A random variable \( X \) has the logistic distribution with location parameter \( \mu \) and scale parameter \( \sigma > 0 \) if its cumulative distribution function is given by

\[
F(x; \mu, \sigma) = \frac{1}{1 + \exp[-(x - \mu)/\sigma]}, \quad -\infty < x < \infty. \tag{6.1}
\]

The corresponding probability density function is

\[
f(x; \mu, \sigma) = \frac{\exp[-(x - \mu)/\sigma]}{\sigma[1 + \exp[-(x - \mu)/\sigma]]^2}, \quad -\infty < x < \infty. \tag{6.2}
\]

Clearly, the above probability density function is symmetric about the location parameter \( \mu \). From now on the logistic distribution with the probability density function given in (6.2) will be denoted as \( L(\mu, \sigma) \).

An early reference to the use of the logistic function \( h(t) = a(1 + b \exp(-ct))^{-1} \) as a growth curve goes back as far as 1838. Since then its use has been very popular for modeling human and biological population expansion, agricultural production data, and bioassay and quantal response data, to mention a few. A book length account on the logistic distribution and its applications is due to Balakrishnan (1992). Interested readers may also refer to the monographs by Johnson et al. (1995) and Balakrishnan and Nevzorov (2003), for more details and relevant references.

Motivated by the capability of the Logistic distribution to describe satisfactory real data, especially in the case of presence of skewness, several generalizations of it
have been suggested which are more flexible than the classical model, with obvious aim to establish better fitting to the available data, see e.g., George et al. (1986), Gupta and Kundu (2010), Lagos-Alvarez et al. (2011), Li and De Moor (1997), Olapade (2004), and references therein.

In the present chapter we introduce a generalized Logistic distribution which offers quite remarkable adaptability in real data arising in financial markets. The key point for the creation of our generalized model is the well-known property of the Logistic distribution that the logit transformation of its cumulative distribution function, i.e. \( \log(F(x))/(1-F(x)) \) is a linear function of \( x \). A plausible generalization of it would arise quite naturally by considering a family of distributions having respective logit transformation of Polynomial type. Apparently, when the need for fitting real data in a probability model arises, the new family is a more powerful choice than \( L(\mu, \sigma) \) since there exist more than two parameters which can be appropriately adjusted to improve the fit.

To further illustrate this point, let us look at Figure 6.1 which displays the histogram of the exchange rates of Euro/CAD for the period 1/4/1999-12/31/2011. It is clear that we are dealing with a bimodal distribution and therefore the \( L(\mu, \sigma) \) model is expected to result in a very inaccurate fit.

**Figure 6.1.** Histogram of the exchange rate data.

This is clearly conveyed by the shape of the logit transformation of the empirical cumulative distribution function, as seen in Figure 6.2, which is far away from being linear as would be expected under the \( L(\mu, \sigma) \) model.
None the less, it seems plausible to search for an appropriate Polynomial function (of degree greater than 1) that could describe satisfactorily the logit transformation; needless to say, having succeeded in the last task, the shape of the histogram would also be reproduced by the generalized distribution associated to the fitted polynomial. These arguments are clearly illustrated in Figure 6.3 and Figure 6.4. We shall further discuss the fitting of these data in Section 6.6.

The organization of the present chapter is as follows: In Section 6.2 we introduce the Polynomial–Logistic distribution which includes the classical Logistic distribution as a special case and illustrate its capability to accommodate a variety of models (unimodal, bimodal, symmetric and non–symmetric).
In Section 6.3 we study the characteristics of the Polynomial-Logistic family of distributions. In Section 6.4 we establish several results pertaining to the order statistics of a random sample for the Polynomial–Logistic distribution, while Section 6.5 offers several expressions for the moments of this family. Finally, in Section 6.6 we exploit the flexibility offered by the Polynomial–Logistic family of distributions to model the Euro foreign exchange reference rates provided by European Central Bank for 6 major currencies (Canadian Dollar, US Dollar, Australian Dollar, Swiss Franc, Japanese Yen, and UK Pound) for the period 1/4/1999-12/31/2011.

6.2. THE POLYNOMIAL–LOGISTIC DISTRIBUTION

Let \( r \) be a positive odd integer and \( a_0, a_1, \ldots, a_r \) real numbers with \( a_r > 0 \), such that the polynomial of degree \( r \)

\[
P(x) = \sum_{i=0}^{r} a_i x^i
\]  

(6.3)

is increasing in \((-\infty, \infty)\). Since \( \lim_{x \to -\infty} P(x) = +\infty \), \( \lim_{x \to -\infty} P(x) = -\infty \) and \( P(x) \) is increasing, it is clear that
\[ F(x) = F(x; a_0, a_1, \ldots, a_r) = \frac{1}{1 + \exp(-\sum_{i=0}^{r} a_i x^i)} = \frac{1}{1 + \exp(-P(x))}, \quad -\infty < x < \infty \]  

(6.4)

will also be an increasing function with

\[ \lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1. \]  

(6.5)

Therefore \( F(x) \) defines a proper cumulative distribution function with support \((-\infty, \infty)\) and corresponding probability density function

\[ f(x) = f(x; a_0, a_1, \ldots, a_r) = \frac{P'(x) \exp(-P(x))}{[1 + \exp(-P(x))]^2}, \quad -\infty < x < \infty. \]  

(6.6)

From now on the distribution defined by (6.4) or equivalently by (6.6) will be called Polynomial–Logistic distribution with parameters \( a_0, a_1, \ldots, a_r \) and will be denoted by \( PL(a_0, a_1, \ldots, a_r) \). For a random variable \( X \) following this distribution we shall write \( X \sim PL(a_0, a_1, \ldots, a_r) \).

It is of interest to note that the probability density function (6.6) can be expressed in terms of the cumulative distribution function (6.4) as follows

\[ f(x) = P'(x) F(x)(1 - F(x)) \]  

(6.7)

In the special case where \( P(x) = x^r \) (\( r \) is a positive odd integer), the resulting Polynomial–Logistic distribution will have cumulative distribution function and probability density function

\[ F(x) = \frac{1}{1 + e^{-x}}, \quad f(x) = \frac{r x^{r-1} e^{-x}}{(1 + e^{-x})^2}, \quad -\infty < x < \infty \]  

(6.8)

respectively and will be referred as Power–Logistic distribution (to be denoted by \( PL_r \)). We shall then write \( X \sim PL_r \).

It is clear that, if \( X \sim PL_r \) then the random variable \( Y = \mu + \sigma X \) will follow a distribution with cumulative distribution function

\[ F_Y(y; \mu, \sigma) = F\left(\frac{x - \mu}{\sigma}\right) = \frac{1}{1 + \exp\left[-\left(\frac{x - \mu}{\sigma}\right)^r\right]}, \quad -\infty < y < \infty \]  

(6.9)

and respective probability density function
\[ f_y(y : \mu, \sigma) = \frac{1}{\sigma} f\left( \frac{x - \mu}{\sigma} \right) = \frac{r(y - \mu)^{r-1} \exp\left[ -\left( \frac{x - \mu}{\sigma} \right)^r \right]}{\sigma^r \left[ 1 + \exp\left( -\left( \frac{x - \mu}{\sigma} \right)^r \right) \right]^2}, \quad -\infty < y < \infty. \] (6.10)

The location–scale family defined by (6.9), (6.10), to be denoted by \( PL_r(\mu, \sigma) \), in the special case \( r = 1 \) reduces to the well-known Logistic distribution with location parameter \( \mu \) and scale parameter \( \sigma \). For \( \mu = 0, \sigma = \sqrt[3]{3}/\pi \) the special case \( PL_1(\mu, \sigma) \) yields the standardized form of the Logistic distribution with probability density function

\[ f_y(y : 0, \sqrt[3]{3}/\pi) = \frac{\pi}{\sqrt[3]{3}} \frac{\exp(-\pi y / \sqrt[3]{3})}{\left[ 1 + \exp(-\pi y / \sqrt[3]{3}) \right]^2}, \quad -\infty < y < \infty. \] (6.11)

In this case we have \( E(Y) = 0, \ Var(Y) = 1. \)

It is well known that the classical Logistic distribution offers an easy to use probability model for data analysis when the available data are skewed; we recall that although several skewed distribution functions exist with support on positive real axis, not many skewed distributions are available with support covering the whole real line. The Polynomial–Logistic distribution introduced in (6.3) - (6.6) offers a very flexible probability model, defined on the whole real line, which can be used for data fitting not only for unimodal skewed data but for bimodal data as well.

In Figure 6.5 we picture the probability density function of the Polynomial–Logistic distribution for \( r = 3 \) and several choices of the polynomial \( P(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \) (note that the condition \( a_2^2 \leq 3a_1a_3 \) should be satisfied so as \( P(x) \) be increasing in \( (-\infty, \infty) \)). As clearly illustrated in the plots, the spectrum of shapes obtained is quite wide: unimodal and bimodal densities, symmetric and non-symmetric can be created by appropriate selection of the parameters \( a_0, a_1, a_2, a_3 \).
Figure 6.5. Probability density function plots of the Polynomial–Logistic distribution

\[ P(x) = x^3 + 10x \]

\[ P(x) = x^3 - 1 \]

\[ P(x) = x^3 + x \]

\[ P(x) = x^3 + x^2 + x + 1 \]
6.3. BASIC CHARACTERISTICS AND PROPERTIES OF THE POLYNOMIAL-LOGISTIC DISTRIBUTION

A well known characteristic of the classical Logistic – distribution is that the logit transformation of its cumulative distribution function is a linear function of $x$. The following proposition shows that the Polynomial–Logistic distribution may accommodate more general cases where the logit transformation departs from linearity.

**Proposition 6.3.1.** If $X \sim PL(a_0, a_1, ..., a_r)$ then the logit transformation of the cumulative distribution function $F(x) = P(X \leq x)$ is a polynomial function of $x$ of degree $r$. More specifically, we have

$$\log \frac{F(x)}{1 - F(x)} = \sum_{i=0}^{r} a_i x^i \quad (6.12)$$

**Proof.** Results immediately from (6.4), on observing that

$$\frac{F(x)}{1 - F(x)} = \exp \left( \sum_{i=0}^{r} a_i x^i \right). \quad (6.13)$$

It is worth mentioning that the LHS of (6.12) can be alternatively interpreted as the negative logarithm of the odd of the survival probability beyond the point $x$, since

$$\log \frac{F(x)}{1 - F(x)} = -\log \frac{1 - F(x)}{F(x)} = -\log \frac{P(A)}{P(A^c)} \quad (6.14)$$

where $A$ stands for the event $\{X > x\}$. Therefore the Polynomial–Logistic model can be used for fitting datasets whose logodds exhibit a polynomial – type behavior. For respective estimation techniques for the Logistic and generalized logistic distributions the interested reader may refer, among others, to Balakrishnan (1990), Li and De Moor (1999), Ragab (1991), Shao (2002), Sreekumar and Thomas (2008) and Zelterman (1987b).

The classical Logistic distribution $PL_1(\mu, \sigma)$ is symmetric about zero and its hazard function (failure rate) is proportional to its cumulative distribution function. The latter property guarantees that the distribution has the IFR (increasing failure rate) property and makes it useful as a growth curve model. The following propositions establish several results of similar nature for the Polynomial–Logistic distribution.
Proposition 6.3.2. If \( P(x) = \sum_{i=1}^{r} a_{2i-1}x^{2i-1} \), \( a_{2i-1} > 0 \) is increasing in \( x \), then the associated Polynomial–Logistic distribution \( PL_{2r-1}(0,a_1,0,a_3,\ldots,0,a_{2r-1}) \) is symmetric about zero.

Proof. It is clear that, in this case we have \( P(-x) = -P(x) \) for every \( x \in (-\infty, +\infty) \).

Therefore
\[
F(-x) = \frac{1}{1 + \exp(-P(-x))} = \frac{1}{1 + \exp(P(x))} = \frac{\exp(-P(x))}{\exp(-P(x)) + 1} = 1 - F(x) \tag{6.15}
\]
and differentiating the identity \( F(-x) = 1 - F(x) \) with respect to \( x \) we obtain
\[
f(-x) = f(x), \quad -\infty < x < +\infty. \tag{6.16}
\]

In view of Proposition 6.3.2, an obvious conclusion to be drawn is that the Polynomial–Logistic distribution (6.8) is symmetric about zero. Moreover the location–scale family \( PL_r(\mu, \sigma) \) defined by (6.9), (6.10) is symmetric around \( \mu \).

Proposition 6.3.3. Let \( X \sim PL_r(a_0, a_1, \ldots, a_r) \). Then

a. The failure rate \( r(t) \) of \( X \) is given by the expression
\[
r(x) = P'(x)F(x) \tag{6.17}
\]
b. The only member of the family \( PL_r(a_0, a_1, \ldots, a_r) \) that has failure rate proportional to its cumulative distribution function is the classical Logistic distribution

c. There exists \( x_0 \in (-\infty, +\infty) \) such that \( X \) has the IFR property for all \( x \geq x_0 \).

Proof. For part (a) it suffices to replace \( f(x), F(x) \) in the formula below
\[
r(x) = \frac{f(x)}{1 - F(x)} \tag{6.18}
\]
by (6.4) and (6.6) respectively, thereof deducing the expression
\[
r(x) = \frac{P'(x)}{1 + \exp(-P(x))} \tag{6.19}
\]
Manifestly, the last expression takes on the more interesting form stated in (6.17).

The conclusion of part (b) is readily justified upon observing that, if the equality \( r(x) = cF(x) \) holds true for all \( x \in (-\infty, +\infty) \), \( c \) is a positive constant, a
straightforward comparison to (6.17) yields $P'(x) = c$ for all $x \in (-\infty, +\infty)$. As a consequence $P(x) = cx + d$ and (6.4) reads

$$F(x) = \frac{1}{1 + \exp(-(cx + d))} = \frac{1}{1 + \exp(-((x - \mu) / \sigma))}, \quad -\infty < x < \infty$$  \hspace{1cm} (6.20)

where $\mu = -d/c$ , $\sigma = 1/c$ and $X$ will follow a classical Logistic distribution with location parameter $\mu$ and scale parameter $\sigma$.

In the special case $r = 1$, part (c) is valid for all $x \in (-\infty, +\infty)$, since if $X \sim PL_t(a_0, a_1)$ we shall have

$$r(x) = (a_0 + a_1 x)' F(x) = a_1 F(x)$$  \hspace{1cm} (6.21)

(as mentioned already in the introduction, this is a characteristic property of the classical Logistic distribution) and $F(x)$ is increasing in $(-\infty, +\infty)$.

For $r \geq 3$, let $x_0$ denote the largest real root of the polynomial $P^r(x)$ (recall that the last polynomial is of degree $r - 2 \geq 1$ and $r - 2$ is an odd integer). For all $x > x_0$ we have $P^r(x) > 0$ and therefore $P'(x)$ will be an increasing function of $x$. The last conclusion may now be used in conjunction with the fact that $F(x)$ is also increasing (in $(-\infty, +\infty)$) to verify that $r(x) = P'(x) F(x)$ is increasing for $x \geq x_0$ \hspace{1cm} $\Box$

Conclusion (c) of Proposition 6.3.3 does not prevent from having members of the Polynomial–Logistic family with the IFR property on the whole real line. One can easily establish a necessary and sufficient condition for the monotonicity of $r(x)$ by simply differentiating the rational expression (6.19). It is straightforward to verify that

$$r'(x) > 0 \iff P^r(x) (e^{P(x)} + 1) + (P'(x))^2 > 0.$$  \hspace{1cm} (6.22)

As a consequence any odd degree polynomial (6.3) satisfying the RHS condition for all $x \in (-\infty, +\infty)$ gives birth to a Polynomial–Logistic distribution which owns the IFR property. In the special case of a Power–Logistic distribution $PL_r$ with $r > 1$, the above condition reduces to $\exp(x') + 1 > -(r/(r - 1))x'$. Manifestly, the last inequality is satisfied for all $x > 0$, therefore the family of Power–Logistic distributions contains distributions which are IFR at least on the positive real axis.

In Figure 6.6 we illustrate the failure rates of the Polynomial–Logistic distribution for several choices of the Polynomial (6.3).
Figure 6.6. Plots of the failure rate for Polynomial–Logistic distributions

\[ P(x) = x^3 + 10x \]

\[ P(x) = x^3 - 1 \]

\[ P(x) = x^3 + x \]

\[ P(x) = x^3 + x^2 + x + 1 \]

In closing this section we mention that the generation of random numbers following a Polynomial–Logistic distribution can be easily accomplished by resorting to the well-known cumulative distribution function inversion technique. More specifically, if \( X \sim PL(a_0, a_1, \ldots, a_r) \), then the random variable
\[ U = F(X) = \frac{1}{1 + \exp(-P(X))} \]  \hspace{1cm} (6.23)

will follow the uniform distribution in the unit interval (0,1). Solving the last equation for \( X \) we deduce

\[ X = P^{-1}\left(\log \frac{U}{1-U}\right) \]  \hspace{1cm} (6.24)

where \( P^{-1} \) denotes the inverse of the (strictly) increasing function \( P(x) \).

Therefore, if \( U \) is a uniformly distributed random variable, the last expression will generate random numbers distributed according to the \( PL(a_0, a_1, \ldots, a_r) \) law. For example, if we wish to generate random numbers following a \( PL_r \) (c.f.(6.8)), the resulting formula will read

\[ X = P^{-1}\left(\log \frac{U}{1-U}\right). \]  \hspace{1cm} (6.25)

### 6.4. MOMENTS

In this section we are providing several formulae for the computation of the moments of the Polynomial–Logistic distribution and offer an explicit expression for the moments of \( PL_r \). Several formulae for the moments of the classical Logistic distribution and its generalizations may be found in the articles contained in Balakrishnan (1992) or in El Saidi et al. (1996).

**Proposition 6.4.1.** If \( X \) follows the Polynomial–Logistic distribution (6.4) with associated polynomial \( P(x) \) then the \( k \)-th moment around zero (provided it exists) is given by the formula

\[
\mu'_k = E(X^k) = k \int_0^\infty x^{k-1} \left( \frac{1}{1 + \exp(P(x))} + \frac{(-1)^{k+1}}{1 + \exp(-P(x))} \right) dx
\]  \hspace{1cm} (6.26)

**Proof.** Write first the moment in the form

\[
\mu'_k = E(X^k) = \int_{-\infty}^{\infty} x^k f(x) dx = \int_{-\infty}^0 x^k F'(x) dx - \int_0^\infty x^k (1 - F(x))' dx
\]  \hspace{1cm} (6.27)

and apply next integration by parts in the integrals of the \( \text{RHS} \) to get the following expressions.
\[
\int_{-\infty}^{0} x^k F'(x)dx = \int_{-\infty}^{0} x^k F(x)dx - k \int_{-\infty}^{0} x^{k-1} F(x)dx = -k \int_{-\infty}^{0} \frac{1}{1+\exp(-P(x))} dx,
\] (6.28)

\[
\int_{0}^{\infty} x^k (1 - F(x))dx = \int_{0}^{\infty} x^k (1 - F(x))\big|_0^\infty = -k \int_{0}^{\infty} x^{k-1} (1 - F(x))dx =
\]

\[
= -k \int_{0}^{\infty} \frac{\exp(-P(x))}{1+\exp(-P(x))} dx.
\] (6.29)

Substituting the last formulae in \( \mu'_k \) we deduce

\[
\mu'_k = -k \int_{-\infty}^{0} x^{k-1} \frac{1}{1+\exp(-P(x))} dx + k \int_{0}^{\infty} x^{k-1} \frac{\exp(-P(x))}{1+\exp(-P(x))} dx
\] (6.30)

and (6.26) is easily deduced on noting that

\[
\int_{-\infty}^{0} x^{k-1} \frac{\exp(-P(x))}{1+\exp(-P(x))} dx = (-1)^k \int_{0}^{\infty} x^{k-1} \frac{\exp(-P(x))}{1+\exp(-P(-x))} dx,
\] (6.31)

\[
\exp(-P(x)) \frac{1}{1+\exp(-P(-x))} = \frac{1}{1+\exp(P(x))}.
\] (6.32)

\[\square\]

It is of interest to note that, following an exact parallel to the proof given above, one can readily verify that the absolute moments \( E(|X|^k) \), \( k = 1,2,... \) of the Polynomial–Logistic distribution can be expressed as

\[
E(|X|^k) = k \int_{0}^{\infty} x^{k-1} \left( \frac{1}{1+\exp(P(x))} + \frac{1}{1+\exp(-P(-x))} \right) dx
\] (6.33)

**Corollary 6.4.1.** If \( P(x) = \sum_{i=1}^{r} a_{2r-1} x^{2r-1}, \ a_{2r-1} > 0 \) is increasing in \( x \) then the moments and absolute moments around zero of \( X \sim PL_{2r-1}(0,a_1,0,a_2,...,0,a_{2r-1}) \) are given by

\[
\mu'_k = E(X^k) = \begin{cases} 0, & \text{if } k \text{ is even} \\ 2k \int_{0}^{\infty} \frac{x^{k-1}}{1+\exp(P(x))}, & \text{if } k \text{ is odd} \end{cases}
\] (6.34)

and

\[
E(|X|^k) = 2k \int_{0}^{\infty} \frac{x^{k-1}}{1+\exp(P(x))} dx
\] (6.35)

**Proof.** In this case we have \( P(-x) = -P(x) \) for all \( x \in (-\infty, +\infty) \) and (6.26), (6.33) reduce to

\[
\mu'_k = k \int_{0}^{\infty} \frac{(1+(-1)^k)}{1+\exp(P(x))} dx,
\] (6.36)
respectively. The first formula yields (6.34) by considering two cases for \( k \) (odd or even) while the second one is exactly (6.35).

**Corollary 6.4.2.** If \( xP(x) > 0 \) for all \( x \neq 0 \) then the \( k \)-th moment around zero of the Polynomial–Logistic distribution (6.4) is given by the following formula

\[
\mu'_k = E(X^k) = k \sum_{j=1}^{\infty} (-1)^{j-1} \int_0^\infty x^{k-1} \left( e^{-jP(x)} - (-1)^{j+1} e^{jP(-x)} \right) dx.
\]  

(6.38)

**Proof.** Since \( xP(x) > 0 \) we conclude that \( |\exp(-P(x))| < 1 \) and \( |\exp(P(-x))| < 1 \) for all \( x > 0 \). Consequently, we may consider the power series expansions.

\[
\frac{1}{1 + \exp(P(x))} = \frac{\exp(-P(x))}{1 + \exp(-P(x))} = \sum_{j=1}^{\infty} (-1)^{j-1} \exp(-jP(x)), \quad x > 0
\]

(6.39)

\[
\frac{1}{1 + \exp(-P(-x))} = \frac{\exp(P(-x))}{1 + \exp(P(-x))} = \sum_{j=1}^{\infty} (-1)^{j-1} \exp(-jP(-x)), \quad x < 0
\]

(6.40)

and the proof is easily completed by imputing them in formula (6.26). \( \Box \)

Manifestly, if \( k \) is an odd integer, \( P(x) = \sum_{i=1}^{r} a_{2i-1} x^{2i-1} \) and \( xP(x) > 0 \) for all \( x \neq 0 \), Corollary 6.4.2 reproduces the result \( \mu'_k = 0 \) (as seen in Corollary 6.4.1), while for \( k \) even we obtain the expression

\[
\mu'_k = E(X^k) = 2k \sum_{j=1}^{\infty} (-1)^{j-1} \int_0^\infty x^{k-1} e^{-jP(x)} dx.
\]

(6.41)

Let us next consider the family of Power–Logistic distributions \( PL_r \) with probability density function and cumulative distribution function given by (6.8). Since \( P(x) = x^r \) (\( r \) a positive odd integer) satisfies the assumptions of Corollary 6.4.1 we may exploit it to establish the following closed expression for the moments of the associated distribution.

**Corollary 6.4.3.** If \( X \) follows the Power–Logistic distribution \( PL_r \), then

\[
E(X^k) = \begin{cases} 
0, & \text{if } k \text{ is odd} \\
2(1 - 2^{-k/r}) j(k/r) + 1, & \text{if } k \text{ is even}
\end{cases}
\]

(6.42)

where \( j(p), \ p > 0 \) is Riemann’s zeta function.
Proof. If \( k \) is even, performing the change of variable \( x'' = t \) in part 2 of (6.34) leads to the expression

\[
E(X^k) = \frac{2k}{r} \int_0^{1+e^t} t^{k/r-1} dx
\]

and exploiting formula (9.513.1) of Gradstheyn and Ryzhik (2007) we obtain

\[
E(X^k) = \frac{2k}{r} (1 - 2^{1-k/r}) \Gamma\left(\frac{k}{r}\right) j\left(\frac{k}{r}\right) = 2(1 - 2^{1-k/r}) \Gamma\left(\frac{k}{r} + 1\right) j\left(\frac{k}{r}\right).
\]

It is noteworthy that, a similar reasoning as above may be used to derive the following explicit formula for the absolute moments of the Power–Logistic distribution

\[
E(|X|^k) = \begin{cases} 
2\log 2, & \text{if } k = r \\
2(1 - 2^{1-k/r}) \Gamma\left(\frac{k}{r} + 1\right) j\left(\frac{k}{r}\right), & \text{if } k \neq r
\end{cases}
\]

for \( k=1,2,\ldots \). We recall that Riemann’s zeta function \( j(p) \) is defined for every \( p > 0 \) and admits the following representation as an infinite series for all \( p > 1 \).

\[
j(p) = \sum_{\infty}^{1} \frac{1}{i^p}
\]

(see e.g. Gradstheyn and Ryzhik (2007)). Since \( j(2) = \pi^2/6 \) and \( j(4) = \pi^4/90 \) we readily obtain the next closed formulae for \( E(|X|^k) \), for \( k = 2r,4r \)

\[
E(|X|^{2r}) = E(X^{2r}) = \frac{\pi^2}{3}, \quad E(|X|^{4r}) = E(X^{4r}) = \frac{7\pi^4}{15}
\]

Table 6.1 depicts the values of the first 5 absolute moments of the Power–Logistic distribution \( PL_r \), \( r=1,3,5,7,9 \). It is of interest to note how fast the absolute moments decrease as \( r \) increases. Needless to say, the case \( r = 1 \) corresponds to the classical Logistic distribution with location and scale parameters 0 and 1 respectively.

**Table 6.1. The first 5 absolute moments of the Power–Logistic distribution \( PL_r \).**

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.3863</td>
<td>3.2899</td>
<td>10.8185</td>
<td>45.4576</td>
<td>233.309</td>
</tr>
<tr>
<td>3</td>
<td>1.0211</td>
<td>1.1486</td>
<td>1.3863</td>
<td>1.7690</td>
<td>2.3646</td>
</tr>
<tr>
<td>5</td>
<td>0.9988</td>
<td>1.0385</td>
<td>1.1149</td>
<td>1.2292</td>
<td>1.3865</td>
</tr>
<tr>
<td>7</td>
<td>0.9946</td>
<td>1.0112</td>
<td>1.0472</td>
<td>1.1018</td>
<td>1.1753</td>
</tr>
<tr>
<td>9</td>
<td>0.9938</td>
<td>1.0013</td>
<td>1.0211</td>
<td>1.0524</td>
<td>1.0948</td>
</tr>
</tbody>
</table>
6.5. ORDER STATISTICS

Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from the Polynomial–Logistic distribution with probability density function and cumulative distribution function as given in (6.4) and (6.6). Denote by \( X_{1:n}, X_{2:n}, \ldots, X_{n:n} \) the order statistics obtained by arranging the sample in ascending order of magnitude. The cumulative distribution function of \( X_{i:n}, 1 \leq i \leq n \) and the respective probability density function are given by (see e.g. Arnold \textit{et al.} (1992) or David and Nagaraja (2003))

\[
F_{i:n}(x) = \sum_{j=i}^{n} \binom{n}{j} (F(x))^{j} (1 - F(x))^{n-j},
\]

\[
f_{i:n}(x) = \frac{dF_{i:n}(x)}{dx} = \frac{n!}{(i-1)!(n-i)!} (F(x))^{i-1} (1 - F(x))^{n-i} f(x).
\]

The distribution and several inferential aspects pertaining to order statistics of the Logistic distribution and its generalizations have been studied by numerous authors; see e.g. Balakrishnan and Leung (1988a,b), Balakrishnan and Malik (1987), Gupta and Balakrishnan (1992) and Zelterman (1987a).

Replacing \( F(x) \) in the first formula by the aid of (6.4) yields

\[
F_{i:n}(x) = \sum_{j=i}^{n} \binom{n}{j} \left( \frac{1}{1 + \exp(-P(x))} \right)^{j} \left( \frac{\exp(-P(x))}{1 + \exp(-P(x))} \right)^{n-j},
\]

and straightforward algebraic manipulations lead to the following more interesting expression

\[
F_{i:n}(x) = \frac{1}{[1 + \exp(P(x))]^n} \sum_{j=i}^{n} \binom{n}{j} \exp(jP(x))
\]

The probability density function \( f_{i:n}(x) \) given above, by virtue of (6.7), takes on the form

\[
f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} P'(x)(F(x))^i (1 - F(x))^{n-i-1}
\]

and again by substitution of \( F(x) \) and reduction we gain the next expression

\[
f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} P'(x) \frac{\exp(iP(x))}{[1 + \exp(P(x))]^{n+1}}
\]

**Proposition 6.5.1.** The probability density function \( f_{i:n}(x) \) of the \( i \)-th order statistic for a random sample of size \( n \) from the Polynomial–Logistic distribution satisfies the recurrence relation
\[ f_{i,n}(x) = \frac{n-i}{i} \exp(P(x)) f_{i-1,n}(x), \quad 1 \leq i < n. \]  

**Proof.** Follows immediately from (6.53). \(\Box\)

The above simple recurrence scheme can be used for an efficient computation of the probability density function of the order statistic in conjunction with the initial condition (c.f.(6.53))

\[ f_{1,n}(x) = nP'(x) \frac{\exp(P(x))}{[1 + \exp(P(x))]^{n+1}}. \]  

**Proposition 6.5.2.** If \( X_1, X_2, \ldots, X_n \) is random sample of size \( n \) from the Polynomial–Logistic distribution with probability density function and cumulative distribution function as given in (6.6) and (6.4) then

\[ \frac{1}{n} \log \frac{P(\min_{1 \leq i \leq n} X_i \leq x)}{P(\max_{1 \leq i \leq n} X_i > x)} = P(x) \]  

**Proof.** Since

\[ F_{1,n}(x) = 1 - (1 - F(x))^n = 1 - \frac{\exp(-nP(x))}{[1 + \exp(-P(x))]^n} \]  

\[ F_{n,n}(x) = (F(x))^n = \frac{1}{[1 + \exp(-P(x))]^n} \]

it follows that

\[ F_{1,n}(x) = 1 - \exp(-nP(x)) F_{n,n}(x) \]  

and therefore

\[ \frac{1 - F_{1,n}(x)}{F_{n,n}(x)} = \exp(-nP(x)). \]

The desired result is now easily derived on noting that

\[ 1 - F_{1,n}(x) = P(\min_{1 \leq i \leq n} X_i > x), \quad F_{n,n}(x) = P(\max_{1 \leq i \leq n} X_i \leq x) \]  

\(\Box\)

It is of interest to note that the outcome of Proposition 5.2 can be exploited for the estimation of the parameters of a Polynomial–Logistic distribution in case we are in possession of independent random samples from this distribution. More precisely, using these samples we may compute the proportion of samples for which the events \( \{\max_{1 \leq i \leq n} X_i \leq x\} \) and \( \{\min_{1 \leq i \leq n} X_i > x\} \) occur, for several values of \( x \). Based on these, the
LHS of (6.56) can be estimated and the coefficients of the Polynomial $P(x)$ may subsequently be computed by fitting a polynomial of order $r$ in the points produced by the above procedure.

Should we be interested in fitting a location–scale Power-Logistic distribution $PL_r(\mu, \sigma)$ in our data ($r$ is a pre-specified odd positive integer and $\mu$ and $\sigma > 0$, unknown parameters), the application of Proposition 5.2 reduces to

$$
\left( \frac{x - \mu}{\sigma} \right)^r = \frac{1}{n} \ln \frac{P(\max X_i \leq x)}{P(\min X_i > x)} = \frac{x - \mu}{\sigma} = \left( \frac{1}{n} \ln \frac{P(\max X_i \leq x)}{P(\min X_i > x)} \right)^{1/r}.
$$

(6.62)

Hence, in this case, one needs to estimate the RHS by the available data and then fit a straight line (through linear regression) to the points generated by the data.

The next propositions offer some recurrence relations satisfied by the moments of the order statistics of a sample drawn from the Polynomial–Logistic distribution.

**Proposition 6.5.3.** Let $\mu_{i_n}^{(k)} = E(X_n^k)$, $1 \leq i \leq n$ and $k \geq 1$ be the $k$-th order moment around 0 for the $i$-th order statistic of a random sample of size $n$ from the Polynomial–Logistic distribution with probability density function and cumulative distribution function as given in (6.6) and (6.4). Then the following recurrences hold true:

a. $\mu_{i_n}^{(k)} = n \sum_{j=1}^{r} \frac{j}{k+j} a_j [\mu_{i_n}^{(k+j)} - \mu_{i_n}^{(k+j+1)}]$  

(6.63)

b. $\mu_{i_n}^{(k)} = i(n-i+1) \sum_{j=1}^{n-1} \frac{j}{k+j} a_j [\mu_{i_n}^{(k+j)} - \mu_{i_n}^{(k+j+1)}]$ for $1 \leq i < n$.

(6.64)

**Proof.** We have that

$$
\mu_{i_n}^{(k)} = E(X^k) = \int_{-\infty}^{\sigma} x^k f_{i_n}(x) dx
$$

(6.65)

and replacing $f_{i_n}(x)$ by (6.52) and $P(x)$ by $P'(x) = \sum_{j=1}^{r} ja_j x^{j-1}$, we get

$$
\mu_{i_n}^{(k)} = \frac{n!}{(i-1)!(n-i)!} \sum_{j=1}^{r} ja_j \int_{-\infty}^{\sigma} x^{k+j-1}(F(x))'(1-F(x))^{n-i} dx
$$

(6.66)

a. For $i=1$, formula (6.66) reduces to

$$
\mu_{i_n}^{(k)} = n \sum_{j=1}^{r} ja_j \int_{-\infty}^{\sigma} x^{k+j-1} F(x)(1-F(x))^r dx,
$$

(6.67)

and, upon integrating by parts, leads to the following equality.
\[ \mu_{tn}^{(k)} = n \sum_{j=1}^{e} ja_j \frac{1}{k + j} \left( - \int_{-\infty}^{\infty} x^{k+j} f(x)(1 - F(x))^n \, dx \right) + n \int_{-\infty}^{\infty} x^{k+j} F(x)(1 - F(x))^{n-1} f(x) \, dx \]  \hspace{1cm} (6.68)

By virtue of (6.52) for \( i = 1 \) and (6.7) we deduce

\[ \int_{-\infty}^{\infty} x^{k+j} f(x)(1 - F(x))^n \, dx = \int_{-\infty}^{\infty} x^{k+j} P'(x) F(x)(1 - F(x))^{n+1} \, dx = \]

\[ = \frac{1}{n+1} \int_{-\infty}^{\infty} x^{k+j} f_{tn+1}(x) \, dx = \frac{1}{n+1} \mu_{tn+1}^{(k+j)}, \]  \hspace{1cm} (6.69)

\[ \int_{-\infty}^{\infty} x^{k+j} F(x)(1 - F(x))^{n-1} f(x) \, dx = \int_{-\infty}^{\infty} x^{k+j} P'(x) F^2(x)(1 - F(x))^n \, dx = \]

\[ = \int_{-\infty}^{\infty} x^{k+j} P'(x) F(x)(1 - F(x))^n \, dx - \int_{-\infty}^{\infty} x^{k+j} P'(x) F(x)(1 - F(x))^{n+1} \, dx = \]

\[ = \frac{1}{n} \mu_{tn}^{(k+j)} - \frac{1}{n+1} \mu_{tn+1}^{(k+j)} \]  \hspace{1cm} (6.70)

and substituting these expressions in (6.66) we readily obtain

\[ \mu_{tn}^{(k)} = n \sum_{j=1}^{e} \frac{j}{k + j} a_j [\mu_{tn}^{(k+j)} - \mu_{tn+1}^{(k+j)}]. \]  \hspace{1cm} (6.71)

b. Applying integration by parts in the integral inside the summation of (6.66) yields

\[ \int_{-\infty}^{\infty} x^{k+j-1} (F(x))' (1 - F(x))^{n-i+1} \, dx = \frac{1}{k + j} \left( -i \int_{-\infty}^{\infty} x^{k+j-1} (F(x))^{i-1} f(x)(1 - F(x))^{n-i+1} \, dx + \right) \]

\[ + (n-i+1) \int_{-\infty}^{\infty} x^{k+j} (F(x))' (1 - F(x))^{n-i} f(x) \, dx \]  \hspace{1cm} (6.72)

The first integral in the RHS equals (recall formula (6.52))

\[ \int_{-\infty}^{\infty} x^{k+j} P'(x)(F(x))' (1 - F(x))^{n-i+2} \, dx = \frac{(i-1)!(n-i+1)!}{(n+1)!} \int_{-\infty}^{\infty} x^{k+j} f_{tn+1}(x) \, dx = \]

\[ = \frac{(i-1)!(n-i+1)!}{(n+1)!} \mu_{tn+1}^{(k+j)} \]  \hspace{1cm} (6.73)

while the second one

\[ \int_{-\infty}^{\infty} x^{k+j} P'(x)(F(x))^{i+1} (1 - F(x))^{n-i+1} \, dx = \frac{i!(n-i)!}{(n+1)!} \int_{-\infty}^{\infty} x^{k+j} f_{tn+1}(x) \, dx = \]

\[ = \frac{i!(n-i)!}{(n+1)!} \mu_{tn+1}^{(k+j)} \]  \hspace{1cm} (6.74)

The desired recurrence relation is effortlessly obtained by combining all the above formulae.
It is worth mentioning that if \( P(x) = x \) (i.e. \( r = 1 \) and \( a_0 = 0, a_1 = 1 \)) then the recurrences provided by Proposition 6.5.2 reduce to
\[
\mu_{tn}^{(k)} = \frac{n}{k+1} [\mu_{tn}^{(k+1)} - \mu_{tn+1}^{(k+1)}], \quad \mu_{tn}^{(k)} = \frac{1}{k+1} [\mu_{tn+1}^{(k+1)} - \mu_{tn+1}^{(k+1)}]
\] (6.75)
and coincide to the ones established for the classical Logistic distribution \( PL_\ell(0,1) \) (see e.g. Gupta and Balakrishnan (1992) or Chapter 23 in Johnson et al. (1995)).

6.6. STATISTICAL ANALYSIS OF EXCHANGE RATE DATASETS

The data examined in this section include Euro foreign exchange reference rates provided by the European Central Bank and were collected from http://www.ecb.int/stats/exchange/eurofxref/html/index.en.html. More specifically we have used the Euro exchange rates for 6 major currencies (Canadian Dollar, US Dollar, Australian Dollar, Swiss Franc, Japanese Yen, and UK Pound) for the period 1/4/1999-12/31/2011. In this section we present the analysis of exchange rates datasets by fitting several Polynomial–Logistic models and commend their efficiency.

We shall first examine in detail the Canadian Dollar (CAD) exchange rates (EUR/CAD) and then give the final fitting results for the rest of the currencies. The sample mean and variance for the EUR/CAD exchange rate data in the reference period is 1.4854 and 0.0113 respectively (standard deviation: 0.1041) with a minimum and maximum value 1.280 and 1.804 respectively. The sample size was 3330 daily rates, while the number of distinct values observed in the sample was 2162.

Let us first try to fit a location–scale Power–Logistic distribution with cumulative distribution function given in (6.9). If we assume that the exchange rates \( X \) follow \( PL_r(\mu, \sigma) \), (we assume that \( r \geq 1 \) is a fixed positive odd integer), it is obvious that the random variable \( Y = (X - \mu)/\sigma \) will follow the standardized \( PL_r(0,1) = PL_r \) distribution (see (6.8)) and applying Corollary 6.4.3 we obtain
\[
E(X) = \mu + \sigma E(Y) = \mu, \quad Var(X) = \sigma^2 Var(Y) = 2\sigma^2 (1 - 2^{(r-2)/r}) \Gamma((r+2)/r) j(2/r).
\]
Therefore the MOM (method of moments) estimators of the parameters \( \mu \) and \( \sigma^2 \) are
\[ \hat{\mu} = \bar{X}, \quad \hat{\sigma}^2 = \frac{s^2}{2(1-2^{(r-2)/r})\Gamma((r+2)/r)j(2/r)}, \]  

(6.76)

where \( s^2 = 0.0113 \) is the sample variance.

In the first block of Table 6.2 we present, for several choices of \( r \), the estimates \( \hat{\sigma}^2 \) of \( \sigma^2 \), the value of the goodness of fit chi-square statistic (the observed and expected frequencies were evaluated by using 10 bins) and the value of Kolmogorov–Smirnov (K-S) distances between the fitted and empirical distribution functions.

An alternative method for estimating the parameters \( \mu \) and \( \sigma \) results on noticing that, in view of Proposition 6.3.1, if \( X \sim PL_r(\mu, \sigma) \) we have

\[ g(x) = \left( \log \frac{F(x)}{1-F(x)} \right)^{1/r} = \frac{x-\mu}{\sigma}. \]  

(6.77)

As a consequence, we could estimate \( \mu \) and \( \sigma \) following the next 3 steps:

- estimate \( g(x) \), by exploiting the available data,
- fit a straight line to the points \((x, g(x))\) by OLS, thereof obtaining an estimate, say \( \hat{a} \) for the slope of the line and \( \hat{b} \) for its intercept,
- use the formulae \( \hat{\mu} = -\hat{b}/\hat{a}, \hat{\sigma} = 1/\hat{a} \) to estimate \( \mu \) and \( \sigma \).

The second block of Table 6.2 provides the same info as the first block, when the alternative estimation method is practiced. As made clear by comparing the corresponding entries in the two blocks

- the MOM and OLS estimators for \( \mu \) match up to two decimal places.
- the MOM and OLS estimators for \( \sigma^2 \) are quite close.
- the fitting by MOM estimators beats slightly the OLS fitting (according to the K-S statistic value).

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \hat{\mu} )</th>
<th>( \hat{\sigma}^2 )</th>
<th>Chi-square statistic</th>
<th>K-S statistic</th>
<th>( \hat{\mu} )</th>
<th>( \hat{\sigma}^2 )</th>
<th>Chi-square statistic</th>
<th>K-S statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.485</td>
<td>0.0034</td>
<td>790.982</td>
<td>0.125</td>
<td>1.486</td>
<td>0.0033</td>
<td>810.009</td>
<td>0.130</td>
</tr>
<tr>
<td>3</td>
<td>1.485</td>
<td>0.0098</td>
<td>&gt;1000</td>
<td>0.091</td>
<td>1.485</td>
<td>0.0108</td>
<td>&gt;1000</td>
<td>0.116</td>
</tr>
<tr>
<td>5</td>
<td>1.485</td>
<td>0.0108</td>
<td>&gt;1000</td>
<td>0.153</td>
<td>1.484</td>
<td>0.0130</td>
<td>&gt;1000</td>
<td>0.200</td>
</tr>
<tr>
<td>7</td>
<td>1.485</td>
<td>0.0111</td>
<td>&gt;1000</td>
<td>0.187</td>
<td>1.484</td>
<td>0.0140</td>
<td>&gt;1000</td>
<td>0.244</td>
</tr>
<tr>
<td>9</td>
<td>1.485</td>
<td>0.0112</td>
<td>&gt;1000</td>
<td>0.211</td>
<td>1.484</td>
<td>0.0145</td>
<td>&gt;1000</td>
<td>0.273</td>
</tr>
<tr>
<td>11</td>
<td>1.485</td>
<td>0.0113</td>
<td>&gt;1000</td>
<td>0.228</td>
<td>1.484</td>
<td>0.0149</td>
<td>&gt;1000</td>
<td>0.290</td>
</tr>
</tbody>
</table>
However, the fitting of our data to the Power–Logistic distribution does not seem adequate by none of the practiced methods.

Alert readers may easily understand that the distribution fitting to our data could be improved by extending the range of candidate distributions to the whole Polynomial–Logistic family (instead of restricting ourselves only to Power–Logistic models) at the expense of heavier computations in estimating the necessary parameters (coefficients of the polynomial $P(x)$). In this case, the method of moments is difficult to apply due to the absence of closed formulae for the moments of the theoretical distributions. None the less, the OLS method can be effortlessly applied to derive the estimates of the polynomial coefficients and then write down the respective cumulative distribution function and probability density function of the fitted Polynomial-Logistic distribution.

Table 6.3 depicts the chi-square statistic and $K$-S statistics when a Polynomial-Logistic distribution is fitted to the Canadian Dollar exchange rates data. The $p$-value for the chi-square goodness of fit test is also provided for each fit. Apparently, the fit provided by the Polynomial-Logistic model beats by far the fit of the Power-Logistic family (compare the corresponding $K$-S values for same $r$’s).

In view of the $K$-S statistic values, one may state that, when $r \geq 3$, the discrepancy between the fitted and empirical cumulative distribution function is less than 6%, a fact indicating a quite reasonable fit. For $r \geq 7$ the distance between the fitted and empirical cumulative distribution function drops below 2.5%.

It is worth mentioning that, by increasing the degree $r$ of the polynomial $P(x)$ $P(x)$, it is usually feasible to arrive at a fitted distribution which is typically acceptable either by the chi-square or the Kolmogorov-Smirnov goodness of fit tests.

<table>
<thead>
<tr>
<th>$r$</th>
<th>Chi-square statistic</th>
<th>K-S statistic</th>
<th>$p$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>810.009</td>
<td>0.131</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>245.444</td>
<td>0.059</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>208.480</td>
<td>0.044</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>31.439</td>
<td>0.025</td>
<td>0.0002</td>
</tr>
<tr>
<td>9</td>
<td>15.352</td>
<td>0.022</td>
<td>0.0817</td>
</tr>
<tr>
<td>11</td>
<td>13.588</td>
<td>0.021</td>
<td>0.1378</td>
</tr>
<tr>
<td>13</td>
<td>13.884</td>
<td>0.021</td>
<td>0.1265</td>
</tr>
<tr>
<td>15</td>
<td>11.049</td>
<td>0.018</td>
<td>0.2724</td>
</tr>
<tr>
<td>17</td>
<td>9.996</td>
<td>0.018</td>
<td>0.3508</td>
</tr>
</tbody>
</table>
More specifically, from the p-values of the chi-square test given in the last column, it is clear that the null hypothesis of having a good fit, could not be rejected at the 1% and 5% significance level for $r \geq 9$; likewise, the null hypothesis of having a good fit, could neither be rejected at the 10% significance level for $r \geq 11$.

As seen in Figure 6.7 where the histogram of the original data is plotted along with the fitted probability density function using a polynomial of degree $r = 17$, the fit of the Polynomial–Logistic distribution to the real data is quite remarkable.

It is of interest to mention that sometimes it is feasible to establish a good fit by considering non complete polynomials of pre-specified degree. This may be proved beneficial since, by omitting non-significant terms, on one hand we have fewer parameters to estimate and on the other we gain simpler cumulative distribution function and probability density function formulae. For example, for the CAD exchange rate data, by fitting a polynomial of degree 15 containing only the odd powers and the constant (i.e. 10 parameters) the Chi-square statistic becomes 13.9672 while the K-S statistic 0.0207, which are still acceptable and not very far away from the ones achieved by fitting a complete 17 degree polynomial (with 18 parameters).

**Figure 6.7.** Polynomial-Logistic fitting for the CAD exchange rate data: Fitted probability density function and histogram of the data
In Table 6.4 we present Polynomial-Logistic models that exhibit adequate fitting to the Euro foreign exchange reference rates for 6 major currencies (Canadian Dollar (CAD), US Dollar(USD), Australian Dollar(AUD), Swiss Franc (CHF), Japanese Yen (JPY), and UK Pound (GBP)), along with the values of the chi-square and K-S statistics. For each model, we register the degree of the fitted polynomial and the number of parameters used (coefficients of the polynomial including the constant term); needless to say, when the second number exceeds the first by one, the fitted polynomial is a complete one.

<table>
<thead>
<tr>
<th>Currency</th>
<th>Polynomial Degree</th>
<th>Number of parameters</th>
<th>Chi-square statistic</th>
<th>K-S statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAD</td>
<td>15</td>
<td>9</td>
<td>32.2355</td>
<td>0.0229104</td>
</tr>
<tr>
<td></td>
<td>17</td>
<td>10</td>
<td>13.9672</td>
<td>0.020766</td>
</tr>
<tr>
<td></td>
<td>19</td>
<td>11</td>
<td>14.1704</td>
<td>0.0218679</td>
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CHAPTER 7. A GENERALIZED POLYNOMIAL LOGISTIC DISTRIBUTION AND APPLICATIONS

7.1 INTRODUCTION

As already mentioned in Chapter 6, the classical Logistic distribution offers an easy to use probability model for data analysis when the available data are skewed or exhibit heavy tails. For example, statistical analysis has revealed that the normality assumption for various asset returns is not supported by empirical evidence; the literature reports extensively that the distributions of several assets (e.g. stock returns, exchange rates etc.) are usually highly peaked and heavy tailed when compared with normal distributions. Therefore, it is important to develop theoretical distribution models explaining asymmetry and heavy tail phenomena.

Due to the simple form of the probability density function and cumulative distribution function and its remarkable capability to describe adequately financial data as well as data from other applied sciences (e.g demography, agriculture, biostatistics etc.), the logistic distribution has drawn considerable research attention and initiated many interesting extensions and generalizations.

Let \( Z \) a random variable that follows the logistic distribution \( L(\mu, \sigma) \) with location parameter \( \mu \) and scale parameter \( \sigma > 0 \). Then its cumulative distribution function will given by

\[
F_Z(z; \mu, \sigma) = \frac{1}{1 + \exp[-(z - \mu)/\sigma]}, \quad -\infty < z < \infty.
\]  

(7.1)

It is clear that, the random variable \( X = (Z - \mu)/\sigma \) will follow a distribution with cumulative distribution function and probability density function

\[
F_X(x) = \frac{1}{1 + \exp(-x)}, \quad f_X(x) = \frac{\exp(-x)}{[1 + \exp(-x)]^2} \quad -\infty < x < \infty
\]  

(7.2)

respectively. From now on the logistic distribution with the cumulative distribution function and probability density function given in (7.2) will be referred to as the standard logistic distribution and will be denoted by \( L(0,1) \).

In Chapter 6, we introduced and studied a natural generalization of the Logistic distribution which offers quite remarkable adaptability in real data arising in finance.
The key point for the generation of their model was the well-known property of the Logistic distribution that the logit transformation of its cumulative distribution function, i.e. \( \ln(F(x))/(1-F(x)) \) is a linear function of \( x \). Motivated by this property, they considered a probability model, named Polynomial Logistic distribution, whose logit cumulative distribution function transformation is of polynomial type.

In the present chapter we introduce a further generalization of this model by introducing two additional parameters that can be used to control for the asymmetry and the decay of tails. The new family nests a variety of distribution models (unimodal, bimodal, symmetric and non–symmetric) and includes as special cases the generalized logistic distribution which has been extensively studied in the literature (see Dubey (1969), Balakrishnan and Leung (1988a, 1988b)), as well as the Polynomial Logistic distribution mentioned above. Because of its flexibility, it offers a rich family for fitting financial data where the presence of skewness and asymmetry is quite common, and making use of models with heavy tails is indispensable.

The new distribution introduced in the present chapter can be used in several frameworks, including modeling of the extreme minima and maxima daily returns, developing distributional models for describing extremes in environmental studies, as appropriate model for the innovation process in ARCH, GARCH and stochastic volatility models for fitting asset returns etc. However we shall only illustrate here the procedure for developing appropriate distributional model for Euro foreign exchange reference rates and returns as well as for daily stock returns. The application of the new distribution in the rest frameworks will be further explored in future works.

The remainder of the chapter is organized as follows. Section 7.2 gives the definition of the Generalized Polynomial-Logistic distribution (GPL distribution), while Section 7.3 provides several of its properties pertinent to the generation of random variables following this specific parametric model, ageing properties, as well as exact expressions for its moments and absolute moments. In Section 7.4 we discuss several methods for estimating the parameters of the GPL distribution while in Section 7.5 a simulation study is carried out in order to compare the estimation methods detailed in Section 7.4. Finally, in Section 7.7, the usefulness of the new distribution is illustrated on real data sets by showing that it is quite flexible in analyzing financial data (more precisely, daily log-returns on the S&P 500 index and the Euro/CAD exchange rates for specific time periods).
7.2 THE GENERALIZED POLYNOMIAL–LOGISTIC DISTRIBUTION

Due to the simple form of the probability density function and cumulative distribution function of the logistic distribution (c.f (7.1) and its remarkable capability to describe adequately financial data and data from other applied sciences (e.g. demography, agriculture, biostatistics etc.), several generalizations have been proposed in the literature. Four prominent types of generalized logistic distributions are as follows (see e.g. Johnson et. al. (1995), Balakrishnan and Nevzorov (2003), Balakrishnan (1990), (1992)):

Type I: 
\[ f^I(x) = f^I(x; a) = \frac{a \exp(-x)}{[1 + \exp(-x)]^a}, \quad -\infty < x < \infty \quad (a > 0) \]  \( (7.3) \)

Type II: 
\[ f^II(x) = f^II(x; a) = \frac{a \exp(-ax)}{[1 + \exp(-x)]^a}, \quad -\infty < x < \infty \quad (a > 0) \]  \( (7.4) \)

Type III: 
\[ f^III(x) = f^III(x; a) = \frac{\Gamma(2a)}{(\Gamma(a))^2} \frac{\exp(-ax)}{[1 + \exp(-x)]^{2a}}, \quad -\infty < x < \infty \quad (a > 0) \]  \( (7.5) \)

Type IV: 
\[ f^IV(x) = f(x; a, b) = \frac{1}{B(a, b) [1 + \exp(-x)]^{a+b}}, \quad -\infty < x < \infty \quad (a, b > 0). \]  \( (7.6) \)

In the above formulae the symbols \( \Gamma(a) \), \( B(a, b) \) are used for the well-known Gamma and Beta functions respectively, i.e.

\[ \Gamma(a) = \int_0^\infty t^{a-1} e^{-t} \, dt, \]  \( (7.7) \)

\[ B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} \, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \]  \( (7.8) \)

It is very easy to verify that:

- if \( X \) follows the Type II generalized logistic distribution, then \( -X \) follows the Type I generalized logistic distribution,

- the Type III distribution is a special case of the Type IV distribution (for \( a=b \)),

- the Type I distribution arises from Type IV as special case by setting \( b=1 \)

Note also that the standard logistic distribution (7.2) can be deduced as a special case of any of the above distributions by appropriately choosing the parameters involved (\( a=1 \) for Types I, II, III and \( a=b=1 \) for Type IV).

In Chapter 6, exploiting the well-known property that the logit transformation of the logistic cumulative distribution function is linear, we introduced the polynomial
logistic distribution whose logit cumulative distribution function transformation is polynomial and illustrated its remarkable fit for financial data. We shall next establish a further generalization couching on the probability model of Type IV logistic distribution.

Let \( r \) be a positive odd integer and \( a_0, a_1, \ldots, a_r \) real numbers with \( a_r > 0 \), such that the polynomial of degree \( r \)

\[
P(x) = \sum_{i=0}^{r} a_i x^i
\]

is increasing in \((-\infty, \infty)\). We shall say that the random variable \( X \) follows the Generalized Polynomial–Logistic (GPL) distribution with parameters \( a > 0 \), \( b > 0 \) and \( a_0, a_1, \ldots, a_r \) if its probability density function is of the form

\[
f(x) = f(x; a, b; a_0, a_1, \ldots, a_r) = \frac{1}{B(a, b)} \frac{P'(x) \exp(-bP(x))}{[1 + \exp(-P(x))]^{a+b}}, \quad -\infty < x < \infty
\]

The distribution defined by (7.10) will be denoted by \( GPL(a, b; a_0, a_1, \ldots, a_r) \) and for the respective random variable \( X \) we shall write \( X \sim GPL(a, b; a_0, a_1, \ldots, a_r) \).

Alternative notations that will also be used in the sequel (especially when the parameters \( a_0, a_1, \ldots, a_r \) are not explicitly displayed) are \( f_p(x; a, b) \) for the probability density function (7.10) and \( GPL_p(a, b) \) for the respective distribution.

Formula (7.10) defines a proper probability density function, that is

\[
\int_{-\infty}^{\infty} f(x) \, dx = 1
\]

To verify that it suffices to replace \( f(x) \) by (7.10) and then change the variable of integration according to the transformation

\[
(1 + \exp(-P(x)))^{-1} = z
\]

It can then be easily checked that the integral reduces to

\[
\int_{0}^{1} f_z(z) \, dz
\]

where

\[
f_z(z) = \frac{1}{B(a, b)} z^{a-1} (1 - z)^{b-1}, \quad 0 \leq z \leq 1
\]

is the probability density function of a Beta distributed random variable. Hence

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{0}^{1} f_z(z) \, dz = 1
\]

Using exactly the same transformation as above, one may immediately conclude that the cumulative density function of \( X \sim GPL_p(a, b) \) takes on the form
\[ F(x) = F(x; a, b; a_0, a_1, \ldots, a_r) = F_p(x; a, b) = I_{[1+\exp(-P(x))]}^{-1}(a, b) \]  

(7.13)

where \( I_z(a, b) \) denotes the well-known incomplete Beta ratio

\[ I_z(a, b) = \frac{B_z(a, b)}{B(a, b)} = \int_0^z t^{a-1}(1-t)^{b-1} \, dt, \quad 0 \leq z \leq 1 \]  

(7.14)

(which is in fact the cumulative distribution function of a Beta distributed random variable). The integral

\[ B_z(a, b) = \int_0^z t^{a-1}(1-t)^{b-1} \, dt, \quad 0 \leq z \leq 1 \]  

(7.15)

is usually called incomplete Beta function or incomplete Beta integral.

From the definition of \( I_z(a, b) \) it is evident that

\[ I_{1-z}(a, b) = 1 - I_z(b, a), \quad I_z(1, 1) = z, \quad I_z(a, 1) = z^a. \]  

(7.16)

When the parameter \( a \) or the parameter \( b \) is a positive integer, the incomplete Beta ratio \( I_z(a, b) \) admits the following simpler expansions as a finite sum

\[ I_z(a, b) = \sum_{k=0}^{b-1} \binom{a+b-1}{b} z^{a+b-k-1}(1-z)^k, \quad \text{if } b \text{ is a positive integer,} \]  

(7.17)

\[ I_z(a, b) = 1 - \sum_{k=0}^{a-1} \binom{a+b-k-1}{k} z^k (1-z)^{a+b-k-1}, \quad \text{if } a \text{ is a positive integer.} \]  

(7.18)

For the aforementioned properties of the incomplete Beta function and other special functions the interested reader may consult Gradshteyn and Ryzhik (2007).

In view of (7.16) we conclude that the special case \( a = 1, b = 1 \), leads to the following cumulative distribution function

\[ F(x; 1, 1; a_0, a_1, \ldots, a_r) = F_p(x; 1, 1) = [1 + \exp(-P(x))]^{-1} \]  

(7.19)

which is in fact the cumulative distribution function of the Polynomial-logistic distribution studied in Chapter 6. From now on we shall be using the simpler notation \( F_0(x) \) to denote this cumulative distribution function, i.e.

\[ F_0(x) = \frac{1}{1 + \exp(-P(x))}, \quad -\infty < x < \infty. \]  

(7.20)

Exploiting expressions (7.17) and (7.18), it is straightforward to verify that, if \( b \) is a positive integer, then the cumulative distribution function of \( GPL_p(a, b) \) takes on the next more convenient form
\[
F_p(x; a, b) = \frac{1}{(1 + \exp(-P(x))^{a+b-1}} \sum_{k=0}^{b-1} \left( \frac{a+b-k}{k} \right) \exp(-kP(x)) \quad (7.21)
\]

while if \( a \) is a positive integer we may write
\[
F_p(x; a, b) = \frac{1}{(1 + \exp(-P(x))^{a+b-1}} \sum_{k=0}^{b-1} \left( \frac{a+b-k}{k} \right) \exp(kP(x)) . \quad (7.22)
\]

Note that, for \( b=1 \) (in which case we are dealing with an extension of the Type I generalized logistic distribution) the probability density function and cumulative distribution function of \( GPL_p(a,1) \) will be given by the next closed formulae
\[
f_p(x; a, l) = \frac{aP(x) \exp(-P(x))}{[1 + \exp(-P(x))]^{a+1}}, \quad F_p(x; a, l) = \frac{1}{[1 + \exp(-P(x))]^a} = (F_0(x))^a . \quad (7.23)
\]

The family of distributions \( \{(F_0(x))^a, a > 0\} \) generated by the cumulative distribution function \( F_0(x) \) is usually referred to as family of “Lehmann alternatives” of \( F_0(x) \) (see Lehmann (1953), and for more recent references, Gibbons and Chakraborti (2003)). It is worth mentioning that, if \( a=n \) is a positive integer, the random variable \( X \sim GPL_p(n,1) \) has the same distribution as the maximum of a sample of \( n \) iid variables distributed according to \( F_0(x) \).

In Figure 7.1 and Figure 7.2 we depict the probability density function of the GPL distribution for several alternative polynomial specifications \( P(x) \) and parameters \( a, b \). As clearly illustrated in the plots, the spectrum of shapes obtained is quite wide: unimodal and multimodal densities, symmetric and positively/negatively skewed can be generated by appropriate selection of the polynomial and the parameters \( a, b \).

It is worth mentioning at this point that, although several skewed distribution functions exist with support on the positive real axis, not many skewed distributions are available with support covering the whole real line. The GPL distribution offers a very flexible probability model, defined on the whole real line, which can be used for data fitting not only for unimodal skewed data but also for bimodal data.

Combining formulae (7.13) and (7.20) we may express the cumulative distribution function of \( GPL_p(a, b) \) as follows
Figure 7.1. Probability density function plots of the GPL distribution for $a \geq 1$, $b \geq 1$

$P(x) = x^3 + 10x$

$P(x) = x^3$

$P(x) = x^3 + x^2 + x + 1$
Figure 7.2. Probability density function plots of the GPL distribution for \( a \leq 1, \ b \leq 1 \)

\[
P(x) = x^3 + 10x
\]

\[
P(x) = x^3
\]

\[
P(x) = x^3 + x^2 + x + 1
\]
\[ F(x; a, b) = I_{F_0(x)}(a, b) = \frac{1}{B(a, b)} \int_0^{F_0(x)} t^{a-1}(1-t)^{b-1} \, dt. \] (7.24)

In view of this, one may state that \( GPL_x(a, b) \) is a member of the class of beta-generalized distributions with baseline cumulative distribution function \( F_0(x) \). This class of distributions seems to have been discussed for the first time by Eugene et al. (2002) who considered the case where \( F_0(x) \) is the cumulative distribution function of the normal distribution and studied what is known as the beta-normal distribution. After the publication of the work of Jones (2004), this class of distributions received considerable attention.

Gupta and Nadarajah (2004) and Nadarajah and Kotz (2004), (2006) introduced and studied the beta-Normal, beta-Gumbel and beta-exponential distribution, Akinsete et al. (2008) introduced the beta-Pareto distribution, Silva et al. (2010) studied the beta modified Weibull distribution, Pescim et al. (2010) introduced the beta generalized half-normal distribution, Barreto-Souza et al. (2011) presented some results on the beta-Frechet distribution, while Paranaiba et al. (2011) defined and investigated the beta Burr XII distribution. Another distribution that belongs to the same family is the beta logistic distribution, which has been mentioned in Jones (2004) and has been around earlier (see e.g. Brown et al., 2002), although its genesis had not originated by the aforementioned procedure; for another interesting generalization of the beta distribution and its application in finance see McDonald and Xu (1995). Recently, Triantafyllou and Koutras (2014) provided a general expression for the failure rate of the generalized beta-generated class of distributions and explored their ageing properties.

Alexander et al. (2012) introduced the generalized beta-generated distributions by using, instead of (7.24), a more flexible family of distributions. More specifically, the new class of distributions has a cumulative distribution function of the form

\[ F(x; a, b; c) = \frac{1}{B(a, b)} \int_0^{F_0(x)} t^{a-1}(1-t)^{b-1} \, dt = \frac{B_{F_0(x)}(a,b)}{B(a,b)} = I_{F_0(x)}(a,b) \] (7.25)

where \( a, b, c \) are positive shape parameters and \( F_0(x) \) the baseline cumulative distribution function. If \( F_0(x) \) is symmetric about zero then the parameters \( a, b, \) and \( c \)
can be used to govern the skewness and tail weight of the generalized beta-generated distribution. This explains the flexibility of the generalized beta-generated family to fit nicely to real life data.

Apparently, the generalized beta-generated family includes all classical beta-generated distributions as special cases \((c = 1]\). If \(c \neq 1\), then the degree of skewness that can be induced by the baseline distribution increases with the kurtosis of it and decreases with the values of the parameters \(a, b\). For more details on several distributional aspects of the generalized beta-generated family (modality, entropy, tail weights, limiting distributions, etc) and its application to data fitting the reader is referred to Alexander et al. (2012).

It is clear that, on choosing \(F_0(x)\) to be the function defined in (7.20) we can generate a further generalization of \(GPL_p(a, b)\), however we shall not pursue this in the present thesis.

### 7.3 PROPERTIES OF THE GENERALIZED POLYNOMIAL-LOGISTIC DISTRIBUTION

In this section we shall present several properties of the GPL. Let us first discuss the problem of the generation of random numbers following a GPL distribution. In view of (7.24) it is clear that if \(X \sim GPL_p(a, b)\) then the random variables

\[
T = \frac{1}{1 + \exp(-P(X))} = F_0(X), \quad 1 - T = \frac{\exp(-P(X))}{1 + \exp(-P(X))} = 1 - F_0(X) \quad (7.26)
\]

follow the Beta distribution with parameters \((a, b)\) and \((b, a)\) respectively. By resorting to the well-known cumulative distribution function inversion technique, we may state that the random variable \(I_r(a, b) = U\) will follow the uniform distribution in the unit interval \((0, 1)\). Hence \(T = H_{a,b}(U)\) where we denote by \(H_{a,b}(\cdot)\) the inverse function of \(I_z(a, b)\) with respect to \(z\), i.e.

\[
H_{a,b}(t) = z \Leftrightarrow I_z(a, b) = t; \quad (7.27)
\]

note that for \(b=1\) we have, by virtue of (7.16)
\[ H_{a,1}(z) = z^{1/a}. \] (7.28)

By solving the equation \[ H_{a,b}(U) = [1 + \exp(-P(X))]^{-1} \] with respect to \( X \) we obtain

\[ X = P^{-1}\left( \ln \frac{H_{a,b}(U)}{1 - H_{a,b}(U)} \right). \] (7.29)

Therefore, if \( U \) is a uniformly distributed random variable, the last expression will generate random numbers distributed according to the GPL distribution. For example, if we wish to generate random numbers following a \( GPL_p(a,1) \) distribution with probability density function and cumulative distribution function given in (7.23), then, in view of (7.28), we can use the following formula

\[ X = P^{-1}\left( \ln \frac{U^{1/a}}{1 - U^{1/a}} \right). \] (7.30)

Let us next move to the study of the GPL distribution shape. A well-known property of the standard logistic distribution (1.2) is that its probability density function is symmetric around 0. This is not always true for GPL even if \( P(x) = x \), because the shape of its probability density function is affected by the choice of the parameters \( a \) and \( b \) as already shown in Figure 7.1 and Figure 7.2.

Let us first consider the case where

\[ P(-x) = -P(x) \quad \text{for every} \quad x \in (-\infty, +\infty), \] (7.31)

which in fact is equivalent to the assumption that the Polynomial \( P(x) \) includes only even powers of \( x \). Then it is obvious that \( P'(-x) = P'(x) \) for every \( x \in (-\infty, +\infty) \) and using (7.10) we get

\[ f(-x) = \frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)} \frac{P'(-x) \exp(-bP(-x))}{[1 + \exp(-P(-x))]^{a+b}} = \frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)} \frac{P'(x) \exp(bP(x))}{[1 + \exp(P(x))]^{a+b}}. \] (7.32)

Writing the last expression in the form

\[ f(-x) = \exp((b-a)P(x)) \frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)} \frac{P'(x) \exp(-bP(x))}{[1 + \exp(-P(x))]^{a+b}} \] (7.33)

we may establish the following formula

\[ f(-x) = f(x) \exp((b-a)P(x)). \] (7.34)

It is now clear that the only case where the probability density function of the GPL is symmetric around 0 (for polynomials satisfying (7.31)) is if \( a=b \).

Using (7.34) we may deduce the following results about the shape of the GPL.
Proposition 7.3.1. Let \( P(x) = \sum_{r=1}^{(e+1)/2} a_{2r-1} x^{2r-1} \), \( a_r > 0 \) be increasing in \( x \).

a. If \( a = b \) then the GPL distribution is symmetric around 0 and therefore its moments of odd order vanish, i.e.
\[
\mu'_{2k-1} = E(X^{2k-1}) = 0 \text{ for } k = 1, 2, \ldots.
\] (7.35)
b. If \( a < b \) then the probability density function of the GPL distribution satisfies the inequality \( f(-x) > f(x) \) for all \( x > 0 \); moreover its moments of odd order are negative, i.e.
\[
\mu'_{2k-1} = E(X^{2k-1}) < 0 \text{ for } k = 1, 2, \ldots.
\] (7.36)
c. If \( a > b \) then the probability density function of the GPL distribution satisfies the inequality \( f(-x) < f(x) \) for all \( x > 0 \); moreover its moments of odd order are positive, i.e.
\[
\mu'_{2k-1} = E(X^{2k-1}) > 0 \text{ for } k = 1, 2, \ldots.
\] (7.37)

Proof. It is clear that, for the Polynomial \( P(x) \) we have
\[
P(-x) = -P(x) \text{ for every } x \in (-\infty, +\infty)
\] (7.38)
and since \( P(x) \) is increasing in \( x \) we conclude that \( P(x) > P(0) = 0 \) for all \( x > 0 \).

Rewriting formula (7.34) in the form
\[
\frac{f(-x)}{f(x)} = \exp((b-a)P(x))
\] (7.39)
and taking into account that \( P(x) > 0 \) for all \( x > 0 \), we readily conclude that

i. if \( a = b \) then \( f(-x) = f(x) \) for all \( x > 0 \),

ii. if \( a < b \) then \( f(-x) < f(x) \) for all \( x > 0 \),

iii. if \( a > b \) then \( f(-x) < f(x) \) for all \( x > 0 \).

and the proof of the first parts of the assertions (a)-(c) in Proposition 7.3.1 is complete.

For the proof of the second parts note first that the moments of odd order \( \mu'_{2k-1} = E(X^{2k-1}) \) are given by the formula
\[
\mu'_{2k-1} = E(X^{2k-1}) = \int_{-\infty}^{\infty} x^{2k-1} f(x) dx = \int_0^{\infty} x^{2k-1} f(x) dx + \int_0^{\infty} x^{2k-1} f(x) dx \cdot
\] (7.40)
and rewrite the first term of the RHS as
\[
\int_{-\infty}^{0} x^{2k-1} f(x) dx = -\int_{0}^{\infty} x^{2k-1} f(-x) dx
\]  
(7.41)

to arrive at the expression
\[
\mu_{2k-1} = E(X^{2k-1}) = \int_{0}^{\infty} x^{2k-1} (f(x) - f(-x)) dx.
\]  
(7.42)

The second parts of the assertions (a)-(c) in Proposition 7.3.1 are immediate consequences of (i)-(iii) which describe the sign of the integrand in each of the three cases \((a = b, a < b \text{ and } a > b)\).

Note that, for the case investigated in Proposition 7.3.1 (i.e. when we employ a Polynomial \(P(x)\) having only the terms \(x, x^3, \ldots, x^r\)), the central moments \(\mu_{2k-1} = E[(X - \mu')^{2k-1}], k=1,2,\ldots\) can be expressed as follows
\[
\mu_{2k-1} = \int_{0}^{\infty} (x - \mu)^{2k-1} - (x + \mu)^{2k-1} e^{(b-a)P(x)} f(x) dx
\]  
(7.43)
(the proof follows on exact parallel to the proof of the second part of Proposition 7.3.1). Therefore, in the case \(a \neq b\), when a non-symmetric distribution arises, the amount of its skewness depends on the difference between \(a\) and \(b\) and its sign on the sign of \(a - b\).

The absolute moments of the Generalized Polynomial Logistic
\[
\nu'_k = E(|X|^k) = \int_{-\infty}^{\infty} |x|^k f(x) dx
\]  
(7.44)
in the case when the polynomial \(P(x)\) admits positive values for \(x > 0\) and negative values for \(x < 0\), can be expressed in the form of infinite series as shown in the next Proposition.

**Proposition 7.3.2.** If \(xP(x) > 0\) for all \(x \neq 0\) then the absolute moments of the Generalized Polynomial Logistic distribution are given by the expression
\[
\nu'_k = E(|X|^k) = \frac{k}{\Gamma(a)\Gamma(b)} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(a+b+j)}{j!} c_p(a,b;k,j)
\]  
(7.45)

where
\[
c_p(a,b;k,j) = \frac{1}{b+j} \int_{0}^{\infty} x^{k-1} \exp(-(b+j)P(x)) dx + \frac{1}{a+j} \int_{0}^{\infty} x^{k-1} \exp((a+j)P(-x)) dx
\]  
(7.46)

**Proof.** Following a reasoning analogous to the one exploited in the proof of the second part of Proposition 7.3.1 we may easily gain the expression
\[
\nu'_k = E(|X|^k) = \int_{-\infty}^{\infty} x^k (f(x) + f(-x)) \, dx.
\]  
(7.47)

Since \( P(x) > 0 \) and \( P(-x) < 0 \) for all \( x > 0 \), the terms \( [1 + \exp(-P(x))]^{-(a+b)} \) and \( [1 + \exp(-P(-x))]^{-(a+b)} \) admit the next power-series representations, for all \( x > 0 \)

\[
\frac{1}{[1 + \exp(-P(x))]^{b+x}} = \sum_{j=0}^{\infty} (-1)^j \binom{a+b+j-1}{j} \exp(-jP(x)),
\]  
(7.48)

\[
\frac{1}{[1 + \exp(-P(-x))]^{x+b}} = \frac{\exp[(a+b)P(-x)]}{[1 + \exp(P(-x))]^{x+b}} = \sum_{j=0}^{\infty} (-1)^j \binom{a+b+j-1}{j} \exp(-jP(x)).
\]  
(7.49)

Replacing the above expressions in (7.10) and substituting \( f(x), f(-x) \) in the integral expression (7.47), we derive the next formula

\[
\nu'_k = \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \binom{a+b+j-1}{j} x \times \lim_{\epsilon \to 0} \int_0^\infty x^k P'(x) \exp(- (b+j)P(x)) \, dx + \int_0^\infty x^k P'(x) \exp((a+j)P(x)) \, dx
\]  
(7.50)

Note next that

\[
\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \binom{a+b+j-1}{j} = \frac{\Gamma(a+b+j)}{j! \Gamma(a)\Gamma(b)}
\]  
(7.51)

and apply integration by parts to get

\[
\int_0^\infty x^k P'(x) \exp(- (b+j)P(x)) \, dx = \frac{k}{b+j} \int_0^\infty x^{k-1} \exp(-(b+j)P(x)) \, dx,
\]  
(7.52)

\[
\int_0^\infty x^k P'(x) \exp((a+j)P(x)) \, dx = \frac{k}{a+j} \int_0^\infty x^{k-1} \exp(-(a+j)P(x)) \, dx.
\]  
(7.53)

The proof is now completed by replacing the last three formulae in (7.50).

Note that, if \( a = b \) and \( P(-x) = -P(x) \) then the quantity \( c_p(a,b;k,j) \) reduces to

\[
c_p(a,a;k,j) = 2 \frac{2}{a+j} \int_0^\infty x^{k-1} \exp(- (a+j)P(x)) \, dx.
\]  
(7.54)

Also, if the polynomial \( P(x) \) contains only the term of the highest order, i.e.

\[
P(x) = cx', \quad c > 0
\]  
(7.55)
then after straightforward calculations we obtain that \( c_p(a, b; k, j) \) admits a closed form, namely

\[
c_p(a, b; k, j) = \frac{\Gamma(k/r)}{rc^{k/r}} \left\{ \frac{1}{(b + j)^{k+1}} + \frac{1}{(a + j)^{k+1}} \right\}. \tag{7.56}
\]

Therefore, in this case, the absolute moments of the Generalized Polynomial Logistic distribution \( \nu'_k = E(|X|^k) \), \( k = 1, 2, \ldots \) can be computed by evaluating the series

\[
\nu'_k = \frac{\Gamma\left(\frac{k}{r} + 1\right)}{\Gamma(a)\Gamma(b)c^{k/r}} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(a + b + j)}{j!} \left( \frac{1}{(b + j)^{k+1}} + \frac{1}{(a + j)^{k+1}} \right). \tag{7.57}
\]

It is not difficult to verify that, in this case, the ordinary moments \( \mu'_k = E(X^k) \) of the Generalized Polynomial Logistic distribution are given by the infinite series

\[
\mu'_k = \frac{\Gamma\left(\frac{k}{r} + 1\right)}{\Gamma(a)\Gamma(b)c^{k/r}} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(a + b + j)}{j!} \left( \frac{1}{(b + j)^{k+1}} + \frac{(-1)^k}{(a + j)^{k+1}} \right). \tag{7.58}
\]

In Table 7.1 we present the values of \( \nu'_k = E(|X|^k) \) for several choices of \( a, b \) and \( r \) (\( c \) was set 1). It is worth mentioning that for small values of \( k \), the convergence of the power series involved above, is quite slow.

We recall that, in the special case \( a = b \), we have \( \mu'_{2k-1} = 0 \) and \( \mu'_{2k} = \nu'_{2k} \), for all \( k = 1, 2, \ldots \) (see also Proposition 7.3.1). Also, from the formula given above, it is clear that the absolute moments of the Generalized Polynomial Logistic distribution with parameters \( b, a \) are exactly the same with the absolute moments of the Generalized Polynomial Logistic distribution with parameters \( a, b \).

As seen in the previous discussion, it is not easy to derive an explicit expression for the moments of the GPL. However a neat formula can be established for the moments of the random variable \( P(X) \) as shown in the next Proposition.

**Proposition 7.3.3.** If \( X \sim GPL_p(a, b) \) then

\[
E[(P(X))^k] = \frac{1}{\Gamma(a)\Gamma(b)} \frac{d}{dt} \left[ \Gamma(a+t)\Gamma(b-t) \right]_{t=0}, \quad k = 1, 2, \ldots \tag{7.59}
\]
Table 7.1. Absolute moments $v_k^r = E(|X|^r)$, $k = 1, 2, 3, 4, 5, 10$ of the Generalized Polynomial Logistic distribution with $P(x) = x^r$, $r = 1, 3, 5, 15$

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Proof. Since the random variable

$$T = (1 + \exp(-P(X)))^{-1}$$  \hspace{1cm} (7.60)

follows the standard Beta distribution, the following formula will hold true for any real numbers $s, t$ such that $a + t > 0$ and $b + s > 0$:

$$E[T^r(1-T)^s] = \frac{1}{B(a,b)} \int_0^1 t^a (1-t)^b dt = \frac{B(a+t,b+s)}{B(a,b)}. $$ \hspace{1cm} (7.61)

Replacing $T = (1 + \exp(-P(X)))^{-1}$ we gain the following expression
\[
E \left[ \frac{\exp(-sP(X))}{1 + \exp(-P(X))} \right] = \frac{B(a+t,b+s)}{B(a,b)} = \frac{\Gamma(a+t)\Gamma(b+s)\Gamma(a+b)}{\Gamma(a)\Gamma(b)\Gamma(a+b+t+s)} .
\] (7.62)

If we bound \( t \) so that \(-a < t < b\) we may apply the previous formula for \( s = -t \) (in which case it is apparent that the conditions \( a + t > 0 \) and \( b + s > 0 \) are met) to arrive at the next formula

\[
E[\exp(tp(X))] = \frac{\Gamma(a+t)\Gamma(b-t)}{\Gamma(a)\Gamma(b)} .
\] (7.63)

Therefore, we have at hand the moment generating function of \( P(X) \) and the moments of it can be readily deduced by differentiating the RHS of the last formula. Exploiting (7.59) we may easily express the moments of \( P(X) \) in terms of derivatives of the Gamma function or equivalently in terms of the well-known digamma function \( \psi(\cdot) \), which is defined as (see e.g. Gradshteyn and Ryzhik (2007))

\[
\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} .
\] (7.64)

As an illustration, note that, a direct application of (7.59) for \( k = 1 \) yields

\[
E[P(X)] = \frac{1}{\Gamma(a)\Gamma(b)} \left. \frac{d}{dt} \left[ \Gamma(a+t)\Gamma(b-t) \right] \right|_{t=0} = \frac{\Gamma''(a)\Gamma(b) - \Gamma(a)\Gamma''(b)}{\Gamma(a)\Gamma(b)} .
\] (7.65)

and therefore

\[
E[P(X)] = \psi(a) - \psi(b) .
\] (7.66)

By similar calculations we get

\[
E[P^2(X)] = \frac{\Gamma''(a)}{\Gamma(a)} + \frac{\Gamma''(b)}{\Gamma(b)} - \frac{2\Gamma''(a)\Gamma''(b)}{\Gamma(a)\Gamma(b)} .
\] (7.67)

and it can be readily verified that

\[
\text{var}[P(X)] = E[P^2(X)] - (E[P(X)])^2 = \psi'(a) + \psi'(b) .
\] (7.68)

The central moments of \( P(X) \) of order 3 and 4 can also be expressed in terms of the digamma function \( \psi(\cdot) \) as follows (the details are omitted)

\[
E[(P(X) - (E[P(X)]))^3] = \psi''(a) + \psi''(b) ,
\] (7.69)

\[
E[(P(X) - (E[P(X)]))^4] = \psi'''(a) + \psi'''(b) + 3(\psi'(a) + \psi'(b)) .
\] (7.70)

It is worth mentioning that the difference \( \psi(a) - \psi(b) \), for \( a,b > 0 \) can be expressed in the form of an infinite series as follows

\[
\psi(a) - \psi(b) = -\sum_{n=1}^{\infty} \frac{\Gamma(n)b^{n-1}}{n!} \left( \int_0^a x^{n-1} \, dx \right) .
\]
\[
\psi(a) - \psi(b) = \sum_{k=0}^{\infty} \left( \frac{1}{b+k} - \frac{1}{a+k} \right), \quad a, b > 0.
\] (7.11)

If the difference between \(a\) and \(b\) is an integer, the last series reduces to a finite sum by appealing to the next formula which is valid for any positive integer \(n\)
\[
\psi(x + n) = \psi(x) + \sum_{k=0}^{n-1} \frac{1}{x+k}, \quad x > 0.
\] (7.72)

In view of these, it is clear that the moments of \(P(X)\) can also be expressed in the form of infinite series; in addition, they reduce to finite sums, if the difference between \(a\) and \(b\) is an integer.

The failure rate (or hazard function) of a random variable \(X\) with probability density function \(f(x)\) and cumulative distribution function \(F(x)\) is defined as
\[
r(x) = \lim_{t \to 0} P[x < X \leq x + t \mid X > x] = \frac{f(x)}{1 - F(x)}.
\] (7.73)

If \(r(x)\) is increasing in \(x\), we say that the associated distribution has the increasing failure rate (\(IFR\)) property, while a decreasing behavior of \(r(x)\) attributes the decreasing failure rate property (\(DFR\)) to the distribution. The classical Logistic distribution has a failure rate, which is proportional to its cumulative distribution function and therefore owns the \(IFR\) property.

If \(X\) follows a GPL distribution, the failure rate \(r(x)\), by virtue of (7.10) and (7.13), may be expressed in the form
\[
r(x) = P'(x)g(F_0(x))
\] (7.74)

where \(g(z)\) is the function
\[
g(z) = \frac{z^a(1-z)^b}{\int_{z}^{1} t^{a-1}(1-t)^{b-1} dt} = \frac{z^a(1-z)^b}{B(a, b) - B_z(a, b)}, \quad 0 < z < 1.
\] (7.75)

On changing the variable of integration in the denominator from \(t\) to \(s = (1-t)/(1-z)\), the integral takes on the form
\[
\int_{z}^{1} t^{a-1}(1-t)^{b-1} dt = (1-z)^b \int_{0}^{1} (1-s)^{a-1}s^{b-1} ds
\] (7.76)
and \(g(z)\) can be expressed as follows
where $G(s,z)=[1-(1-z)s]^{-1}/z^a$. Straightforward calculations yield

$$\frac{\partial}{\partial z} \ln G(x,z) = -\frac{a(1-z)+zs}{z(1-(1-z)s)} < 0, \quad 0 < z < 1$$

Accordingly, the function $G(s,z)$ decreasing in $z$, a fact revealing that that $g(z)$ is increasing in $z$.

Since $F_0(x)$ is increasing for all $x \in (-\infty, \infty)$, the composite function $g(F_0(x))$ appearing in (7.74) is also increasing in $x$ for all $x \in (-\infty, \infty)$.

It is now obvious that, if $r = 1$ (in which case $P'(x)$ is a positive constant), the failure rate $r(x)$ will be an increasing function of $x$ for all $x \in (-\infty, \infty)$; hence in this special case, the Generalized Polynomial Logistic has the IFR property.

For $r \geq 3$ let us denote by $x > x_0$ the largest real root of the polynomial $P(x)$ (recall that the last polynomial is of degree $r-2 \geq 1$ and $r-2$ is an odd integer; as a consequence, it has at least one real root). Then $P''(x) > 0$ for all $x > x_0$ and therefore $P'(x)$ will be an increasing function of $x$ for all $x > x_0$. The last conclusion may now be used in conjunction with the fact that $g((1+\exp(P(x)/x)^{-1})$ is increasing in $x$ (for all $x \in (-\infty, +\infty)$) to verify that (7.74) is increasing for $x \geq x_0$.

The above discussion does not necessarily mean that the members of the GPL family, for $r \geq 3$ do not satisfy the IFR property on the whole real line. The next Proposition describes a family of GPL distributions that do have the IFR property and a family that have the DFR property.

**Proposition 7.3.4.** Let $X \sim GPL_r(a,b)$.

a. If $a > 1$ and $P(x)$ satisfies the condition

$$P''(x)[1+\exp(P(x))] + (P'(x))^2 > 0 \text{ for all } x \in (-\infty, +\infty),$$

then $X$ has the IFR property.

b. If $0 < a < 1$ and $P(x)$ satisfies the condition
\[ P'(x)[1 + \exp(P(x)) + (P'(x))^2 < 0 \text{ for all } x \in (-\infty, +\infty), \]  

(7.80)

then \( X \) has the DFR property.

**Proof.** Using standard algebraic manipulations, it is easy to verify that the failure rate \( r(x) \) of the GPL distribution can be expressed in the form

\[
 r(x) = \frac{1}{B(a, b)} P'(x) F_0(x) \cdot \frac{(F_0(x))^{a-1}(1 - F_0(x))^b}{1 - I_{F_0(x)}(a, b)} 
\]  

(7.81)

or, on introducing the function

\[
 h(z) = \frac{z^{a-1}(1 - z)^b}{1 - I_z(a, b)} 
\]

(7.82)

in the more compact form

\[
 r(x) = \frac{1}{B(a, b)} \frac{d \ln(1 - F_0(x))}{dx} \cdot h(F_0(x)). 
\]

(7.83)

It is not difficult to prove that the function \( h(z) \) is decreasing for all \( z \in (0, 1) \) if \( a > 1 \) and increasing for all \( z \in (0, 1) \) if \( 0 < a < 1 \). To achieve that it suffices to express \( h(z) \) as follows (c.f. (7.16))

\[
 h(z) = \frac{z^{a-1}(1 - z)^b}{I_{-z}(b, a)} = B(a, b) \int_0^1 t^{b-1}(1 - t)^{a-1} dt 
\]

(7.84)

and then change the variable of integration in the denominator from \( t \) to \( s = t/(1 - z) \),

to arrive at

\[
 h(z) = B(a, b) \int_0^z \left( \frac{1}{1 - z} s \right)^{a-1} s^{b-1} ds = \frac{B(a, b)}{\int_0^z \left( s + \frac{1 - s}{z} \right)^{a-1} s^{b-1} ds}. 
\]

(7.85)

Since, the quantity inside the integral is strictly decreasing in \( z \) for \( a > 1 \) and strictly increasing in \( z \) for \( 0 < a < 1 \) we readily conclude that \( h(z) \) is strictly increasing in \( z \) for \( a > 1 \) and strictly decreasing in \( z \) for \( 0 < a < 1 \).

Since \( F_0(x) \) is increasing for all \( x \in (-\infty, \infty) \), the composite function \( h(F_0(x)) \) will have the same type of monotonicity, i.e.

- \( h(F_0(x)) \) is a strictly increasing function for all \( x \in (-\infty, \infty) \), if \( a > 1 \),
- \( h(F_0(x)) \) is a strictly decreasing function for all \( x \in (-\infty, \infty) \) if \( 0 < a < 1 \).
Note next that
\[ \frac{d}{dx} (P'(x)F_0(x)) > 0 \iff P^*(x)[\exp(P(x)) + 1] + (P'(x))^2 > 0 \] (7.86)
which proves that \( P'(x)F_0(x) \) is an increasing (decreasing) function of \( x \) if and only if
\[ P^*(x)[\exp(P(x)) + 1] + (P'(x))^2 > 0 \quad \text{(7.87)} \]
The results of the Proposition follow immediately by taking into account that the product of positive increasing (decreasing) functions is an increasing (decreasing) function.

As an illustration, consider the polynomial \( P(x) = x^3 + 2x \). It is not difficult to verify that the condition
\[ P^*(x)[1 + \exp(P(x))] + (P'(x))^2 > 0 \] (7.88)
is met for all \( x > 0 \). On the other hand, for \( x < 0 \) we have \( \exp(x^3 + 2x) < 1 \) and we may write
\[ P^*(x)[1 + \exp(P(x))] + (P'(x))^2 = 6x[1 + \exp(x^3 + 2x)] + (3x^2 + 2)^2 > 12x + (3x^2 + 2)^2 \] (7.89)
and the last term is strictly positive for all \( x < 0 \). Hence the required condition on \( P(x) \) is satisfied by all \( x \in (-\infty, +\infty) \) and all the distributions \( GPL_p(a,b) \) with \( a > 1 \) will have the IFR property.

In Figure 7.3 we illustrate the failure rates of the GPL distribution for several choices of the Polynomial \( P(x) \) and the parameters \( a, b \).

### 7.4 STATISTICAL INFERENCE

The use of the Generalized Polynomial Logistic for describing real data sets requires the use of efficient estimators of the model parameters which may be derived by exploiting the available data. Let us first assume that all the parameters \( (a, b) \) and the polynomial coefficients \( (a_0, a_2, \ldots, a_r) \) are unknown. Given the values \( x_1, x_2, \ldots, x_n \) of a sample of size \( n \) from the distribution (7.10), the loglikelihood takes on the following form.
Figure 7.3. Plots of the failure rate for GPL distributions

\[ P(x) = x^3 + 10x \]

\[ P(x) = x^3 \]

\[ P(x) = x^3 + x^2 + x + 1 \]
\[ l(a,b;a_0,a_1,\ldots,a_n) = n(\ln \Gamma(a+b) - \ln \Gamma(a) - \ln \Gamma(b)) + \sum_{i=1}^{n} \ln P(x_i) - \sum_{i=1}^{n} P(x_i) - (a+b)\sum_{i=1}^{n} \ln(1 + \exp(-P(x_i))) \]  
\tag{7.90}

and the maximum likelihood equations read

\[ n\psi(a+b) - n\psi(a) - \sum_{i=1}^{n} \ln (1 + \exp(-P(x_i))) = 0 \]  
\tag{7.91}

\[ n\psi(a+b) - n\psi(b) - \sum_{i=1}^{n} P(x_i) - \sum_{i=1}^{n} \ln (1 + \exp(-P(x_i))) = 0 \]  
\tag{7.92}

\[ \frac{j}{P(x_i)} x_i^{j-1} - b \sum_{i=1}^{n} x_i^j + (a+b) \sum_{i=1}^{n} \frac{\exp(-P(x_i))}{1 + \exp(-P(x_i))} x_i^j = 0, \quad j = 0,1,\ldots,r \]  
\tag{7.93}

It is apparent that the system of \( r + 3 \) equations (7.91) - (7.93) is non-linear, so it can only be solved by resorting to iterative numerical procedures (alternatively, one could use an appropriate software, e.g. SAS (Proc NLMixed) to maximize the log-likelihood function (7.90)).

Since the aforementioned approach may result in a computationally demanding or even intractable scheme we shall discuss the estimation problem in the case where only the parameters \( a,b \) or only the polynomial coefficients need to be estimated (i.e. the polynomial coefficients are known in advance for the first case or the parameters \( a,b \) have been fixed by the user in the second case).

a. Estimation of the parameters \( a,b \) for a given polynomial \( P(x) \)

In this case, only the first two likelihood equations (7.91), (7.92) should be used; therefore \( a \) and \( b \) should be computed by solving the following system

\[ \psi(a+b) - \psi(a) = \frac{1}{n} \sum_{i=1}^{n} \ln(1 + \exp(-P(x_i))) \]  
\tag{7.94}

\[ \psi(a+b) - \psi(b) = \frac{1}{n} \sum_{i=1}^{n} P(x_i) + \frac{1}{n} \sum_{i=1}^{n} \ln(1 + \exp(-P(x_i))) \]  
\tag{7.95}

or equivalently

\[ \psi(a+b) - \psi(a) = \frac{1}{n} \sum_{i=1}^{n} \ln(1 + \exp(-P(x_i))) \]  
\tag{7.96}

\[ \psi(a) - \psi(b) = \frac{1}{n} \sum_{i=1}^{n} P(x_i) \]  
\tag{7.97}

A reasonable estimate of the values of \( a,b \) could be established by exploiting the approximate formula \( \psi(x) \approx \ln(x-0.5) \) which is quite accurate for \( x \geq 1 \) (see Figure 7.4).

Then (7.96), (7.97) reduce to
\[
\frac{2a + 2b - 1}{2a - 1} = \exp\left(\frac{1}{n} \sum_{i=1}^{n} \ln(1 + \exp(-P(x_i)))\right) = A . \tag{7.98}
\]

\[
\frac{2a - 1}{2b - 1} = \exp\left(\frac{1}{n} \sum_{i=1}^{n} P(x_i)\right) = B \tag{7.99}
\]

and the following estimates are deduced for \( a \) and \( b \)

\[
a = \frac{AB - 1}{2(AB - B - 1)} , \quad b = \frac{AB - B}{2(AB - B - 1)} . \tag{7.100}
\]

**Figure 7.4.** Exact (solid line) and approximate values (dotted line) for the psi function.

Note that, in view of (A4.1), if one of the parameters \( a, b \) is a fixed positive integer, the computation of the unknown parameter can be achieved by solving a polynomial equation. For example, if \( b = 2 \), equation (7.96) reduces to

\[
\frac{1}{a} + \frac{1}{a + 1} = e^A \iff a = -\frac{e^A + 2 + \sqrt{e^{2A} + 4}}{2e^A} \tag{7.101}
\]

It is also worth mentioning that in the special case \( b = 1 \) (i.e. when we consider the family of Lehmann alternatives for the GPL distribution), the maximum likelihood estimator of \( a \) reduces to

\[
\hat{a} = \frac{n}{\sum_{i=1}^{n} \ln(1 + \exp(-P(x_i)))} \tag{7.102}
\]

The random variables in the summation in the denominator, have the distribution of \(-\ln T\), where \( T \) is the random variable shown in (7.26). Taking this
into account we conclude that $T$ has now a probability density function of the form $at^{a-1}, \ 0 < t < 1$, and it is not difficult to verify that $\hat{a}$ has the same distribution as $n/G$, where $G$ follows the gamma distribution with parameters $a$ and $n$. Straightforward calculations reveal that

$$E(\hat{a}) = \frac{n}{n-1} a, \quad \text{var}(\hat{a}) = \frac{n^2}{(n-1)^2(n-2)} a^2 \quad (7.103)$$

(for $n > 2$), therefore

$$\bar{a} = \frac{n-1}{n} \hat{a} = \frac{n-1}{\sum_{i=1}^{n} \ln(1 + \exp(-P(x_i)))} \quad (7.104)$$

offers an unbiased estimator of $a$ with variance

$$\text{var}(\bar{a}) = \frac{a^2}{n-2}. \quad (7.105)$$

It goes without saying that $\hat{a}$ and $\bar{a}$ are asymptotically equivalent.

A similar reasoning reveals that, in the same special case $a=1$, the maximum likelihood estimator of $b$ reduces to

$$\hat{b} = \frac{n}{\sum_{i=1}^{n} P(x_i) + \sum_{i=1}^{n} \ln(1 + \exp(-P(x_i)))} \quad (7.106)$$

b. Estimation of the polynomial coefficients when the parameters $a, b$ are known

In this case the method of maximum likelihood requires the solution of the system of $r+1$ equations (7.93), with respect to $a_0, a_1, ..., a_r$; this is quite involved and the numerical solution will be computationally demanding.

A simpler system might be constructed by using the method of moments. More specifically, if we wish to use a complete polynomial of order $r$, the need for estimating $r+1$ parameters arises and this can be achieved by evaluating the quantities appearing in the RHS of (7.59) for $k=1,2,...,r+1$ and replacing the moments $E(X^j)$, $j=1,2,...,r+1$ in the LHS by their sample estimates $m'_j = (1/n) \sum_{i=1}^{n} x'_i$

If the Polynomial $P(x)$ is not complete, we can use (7.59) for as many values of $k$, as the number of the parameters to be estimated. For illustration purposes let us assume that we are interested in fitting a polynomial of order 3, satisfying the condition $P(-x) = -P(x)$ for all $x > 0$; therefore $P(x) = a_3 x^3 + a_1 x$ and according to
Proposition 7.3.1 the values of $a, b$ can be appropriately selected so as the symmetry (for $a = b$) or skewness (for $a \neq b$) of the data is captured by the Generalized Polynomial Logistic model. Since only two parameters (i.e. $a, a_3$) need to be estimated, we may apply (7.59) for $k = 1, 2$ to get (c.f. (7.66) and (7.68))

\[
E[a_2 X + a_3 X^3] = \psi(a) - \psi(b)
\]

and replacing $E(X^j)$ by $m'_j$, $j = 1, 2, 3, 4, 6$ we deduce the system

\[
a_i m'_1 + a_i m'_1 = \psi(a) - \psi(b)
\]

\[
a_i m'_2 + a_i m'_2 + 2a_i a_j m'_1 = \psi'(a) + \psi'(b) + (\psi(a) - \psi(b))^2
\]

which requires the solution of a quadratic equation in $a_i$ (if we solve the first equation in terms of $a_3$ and replace it in the second).

The benefit of using the method of moments instead of the maximum likelihood method is that the former requires the solution of a system of polynomial equations, while the latter for a system of transcendental equations (c.f. (7.93)).

Let us finally discuss a third estimation method based on the method of least squares. Recalling that $H_{a,b}(z)$ denotes the inverse of the function $I_{a,b}(a,b)$ defined in (7.27), formula (7.13) yields

\[
\frac{1}{1+\exp(-P(x))} = H_{a,b}(F(x))
\]

and solving the last equation for $P(x)$ we get

\[
P(x) = \ln \frac{H_{a,b}(F(x_i))}{1 - H_{a,b}(F(x_i))}, \quad i = 1, 2, ..., n.
\]

As a consequence, given the values $x_1, x_2, ..., x_n$ of a sample of size $n$, we could estimate $P(x)$ by the quantities

\[
\hat{P}(x_i) = \ln \frac{H_{a,b}(\hat{F}(x_i))}{1 - H_{a,b}(\hat{F}(x_i))}, \quad i = 1, 2, ..., n
\]

where $\hat{F}$ is the empirical cumulative distribution function of $X$ as estimated from the available data. Then the coefficients of $P(x)$ can be estimated by fitting a polynomial of order $r$ to the points $(x_i, \hat{P}(x_i))$ by the method of OLS.
In the special case \( b = 1 \) (family of Lehmann alternatives (7.23)), in view of (7.24), the estimates (7.112) admit the simpler form

\[
P(x) = \ln \frac{(\hat{F}(x_i))^{u_a}}{1 - (\hat{F}(x_i))^{u_a}}, \quad i = 1, 2, \ldots, n
\]

(7.113)

In closing this section, it should be mentioned that one could combine the methods presented in the present section to obtain estimates of all the parameters \( a, b, a_0, a_i, \ldots, a_r \) by working through the method of trial and error; more specifically, one may start with an initial set of values for \( a, b \) in order to estimate the polynomial coefficients, then re-estimate \( a \) and \( b \) using the estimated polynomial and the loop repeats itself until an appropriate stopping criterion is met.

### 7.5. A SIMULATION STUDY FOR COMPARING THE POLYNOMIAL COEFFICIENTS ESTIMATION METHODS

In this section we compare the three methods described in the previous section for the estimation of the polynomial coefficients when the parameters \( a, b \) are known. To achieve that, we shall carry out a Monte Carlo simulation study with 2000 replications using sample sizes \( n = 100, n = 300 \) and \( n = 500 \). The random number generation was performed using the inversion method, as detailed in Section 3, i.e. we created a random variable \( U \) that follows the uniform distribution in the unit interval \((0,1)\) and then used formula (7.29). We considered the case where the polynomial is of the form \( P(x) = a_3x^3 + a_i x \), so that the baseline distribution is symmetric and let the parameters \( a, b \) take care for the symmetry (for \( a = b \)) or skewness (for \( a \neq b \)) of the GPL distribution; two different polynomials \( P(x) = a_3x^3 + a_i x \) and several choices of \( a, b \) have been practiced.

In each replication a random sample of size \( n \), say \( x_1, x_2, \ldots, x_n \), is drawn from the GPL distribution of order \( r = 3 \) with known parameters \( a, b \) and specific choices of the polynomial coefficients \( a_i, a_3 \). Then the parameters \( a_i, a_3 \) are re-estimated from the sample by three different methods:

a. the method of maximum likelihood which calls for the solution of the system of equations (7.91), (7.92), (7.93), which will now consist of two non-linear
equations with two unknowns $a_1, a_3$. In our study, instead of that we used a numerical method to maximize the log-likelihood function (7.90).

b. the method of least squares which fits a polynomial $P(x) = a_3x^3 + a_1x$ to the points $(x_i, \hat{P}(x_i))$, where $\hat{P}(x_i)$ are calculated by (7.112).

c. the method of moments which requires the computation of the sample moments $m'_j = (1/n)\sum_{i=1}^{n} x_i^j$, $j = 1,2,3,4,6$ and the solution of the system (7.109), as illustrated in the previous section.

Table 7.2 lists the means of the three estimates of the parameters for several models corresponding to different choices of the parameters $a$, $b$, polynomials $P(x) = a_3x^3 + a_1x$ and sample sizes $n$. The choices of $a$ and the polynomials $P(x) = a_3x^3 + a_1x$ cover all the spectrum of shapes encompassed in the GPL family, i.e. unimodal and multimodal densities, symmetric and positively/negatively skewed densities. In Table 7.3 the root mean squared errors (RMSE) are provided for each model contained in Table 7.2.

As clearly indicated from Table 7.2, the quality of estimation improves as the sample size increases. The method of maximum likelihood and the least squares method perform better that the method of moments. The maximum likelihood performs slightly better than the least squares method, however it should be stressed that the calculation of the maximum likelihood estimates requires the use of time consuming iterative numerical techniques since the system of (transcendental) equations to be solved is quite involved. On the other hand the method of moments is the fastest in terms of computing time needed, however in some cases fails miserably to estimate the parameters; moreover in other cases it produces invalid estimates (non-increasing polynomial) or no estimates at all (the quadratic system of equations has complex solutions).

The picture conveyed from Table 7.3 for the RMSE of the three estimation methods is almost the same. The RMSE for the method of maximum likelihood is the smallest among the three while MOM has the worst performance. Note also that for all three methods, the level of the RMSE decays as the sample size increases.
Table 7.2. Mean estimates for the parameters $a_1,a_3$ of the GPL distribution with $P(x) = x^3 + x$ and known parameters $a$ and $b$

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$n$</th>
<th>Mean estimate of $a_1$ by $LS$</th>
<th>Mean estimate of $a_1$ by $ML$</th>
<th>Mean estimate of $a_1$ by $MOM$</th>
<th>Mean estimate of $a_3$ by $LS$</th>
<th>Mean estimate of $a_3$ by $ML$</th>
<th>Mean estimate of $a_3$ by $MOM$</th>
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<tbody>
<tr>
<td>1.5</td>
<td>1.5</td>
<td>100</td>
<td>0.8821</td>
<td>1.0270</td>
<td>2.6299</td>
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<td></td>
<td></td>
<td>300</td>
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<td>1.0130</td>
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<td></td>
<td></td>
<td>500</td>
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<td>2.5433</td>
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<td>1.0000</td>
<td>-2.1971</td>
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<tr>
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Table 7.3. RMSE for the estimation of the parameters $a_1,a_3$ of the GPL distribution

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$n$</th>
<th>RMSE when estimating $a_1$ by $LS$</th>
<th>RMSE when estimating $a_1$ by $ML$</th>
<th>RMSE when estimating $a_1$ by $MOM$</th>
<th>RMSE when estimating $a_3$ by $LS$</th>
<th>RMSE when estimating $a_3$ by $ML$</th>
<th>RMSE when estimating $a_3$ by $MOM$</th>
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<td>0.1296</td>
<td>0.107</td>
<td>0.0853</td>
<td>0.1565</td>
</tr>
</tbody>
</table>

$P(x) = x^3 + x$
7.6 EVALUATION OF VALUE AT RISK AND EXPECTED SHORTFALL

The Value at Risk (VaR) and the Expected Shortfall are two widely used risk measures in financial risk management (Altzner et al. (1999)). A brief presentation of these risk measures has been given in Chapter 2.

If $X$ is a random variable, then the Value-at-Risk at confidence level $0 < a < 1$ is defined as follows

$$\Pr(X \leq -\text{VaR}) = p$$  \hspace{1cm} (7.114)

where $p = 1 - a$ (see equation (1.9) in Chapter 2). In the sequel we shall be using the notation $\text{VaR}_p$, $0 < p < 1$ to indicate the Value-at-Risk at confidence level $a = 1 - p$.

Therefore $\text{VaR}_p$ satisfies the condition

$$\Pr(X \leq -\text{VaR}_p) = p \iff F(-\text{VaR}_p) = p$$  \hspace{1cm} (7.115)

where $F(\cdot)$ denotes the cumulative distribution function of the random variable $X$.

If $X$ follows a Normal distribution with mean $\mu$ and variance $\sigma^2$, then the above condition yields

$$\Pr\left(\frac{X - \mu}{\sigma} \leq \frac{-\text{VaR}_p - \mu}{\sigma}\right) = p \iff \Phi\left(\frac{-\text{VaR}_p - \mu}{\sigma}\right) = p$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard Normal random variable. Therefore, $\text{VaR}_p$ can be evaluated by the following formula

$$\text{VaR}_p = -\mu - \sigma \Phi^{-1}(p) = -\mu + \sigma \Phi^{-1}(a)$$

(compare to formula (2.7) in Chapter 2 where, however, we have assumed that the loss distribution is Normal with mean $\mu$ and variance $\sigma^2$).

Let us now assume that $X$ follows a GPL distribution and that, exploiting the available data, we have estimated all parameters involved in it (i.e. $a$, $b$ and $a_0, a_1, \ldots, a_r$). The question is how can one proceed to the estimation of $\text{VaR}_p$ for given $0 < p < 1$. Manifestly, the condition (7.115), in view of (7.13) and (7.20), takes on the form

$$I_{\text{VaR}_p}(a, b) = p.$$  \hspace{1cm} (7.116)

Applying now (7.110), we may write
\[ H_{a,b}(p) = F_0(-VaR_p) = \frac{1}{1 + \exp(-P(-VaR_p))} \quad (7.117) \]

and solving for \(VaR_p\), we get

\[ VaR_p = -P^{-1}\left(\ln \frac{H_{a,b}(p)}{1 - H_{a,b}(p)}\right). \quad (7.118) \]

For illustrative purposes, let us consider the special case where \(b = 1\) and 
\[ P(x) = a_x x^r + a_0, \quad r \geq 1. \] 
Then, by (7.16), we have
\[ H_{a,b}(p) = p^{1/a} \quad (7.119) \]
and it is evident that
\[ P^{-1}(x) = \left(\frac{x - a_0}{a_r}\right)^{1/r}. \quad (7.120) \]

Direct substitution of (7.119) and (7.120) in (7.118) yields the next closed formula for \(VaR_p\).

\[ VaR_p = \left[\frac{1}{a_r} \left(\ln \frac{p^{1/a} - a_0}{1 - p^{1/a} - a_0}\right)^{1/r}\right]. \quad (7.121) \]

In Table 7.5 we plot \(VaR_p\) as a function of \(p\) for several choices of the Polynomial \(P(x)\) and the parameters \(a,b\).

**Figure 7.5. Plot of \(VaR_p\) as a function of \(p\) for the GPL distribution**

\[ P(x) = x^3 + x^2 + x + 1 \]

Let us next turn our attention to the evaluation of the Expected Shortfall. Recall that Expected Shortfall \(ES(x_0)\) at a specific point \(x_0 \in (-\infty, \infty)\) is defined as
\( ES(x_0) = E(X|X < x_0) \) \hspace{1cm} (7.122)

(see formula (1.15) in Chapter 2). The above formula can be rewritten in terms of the cumulative distribution function \( F(x) \) as follows

\[
ES(x_0) = \frac{\int_{x_0}^{x} x f(x)dx}{\Pr(X < x_0)} = \frac{\int_{x_0}^{x} x F'(x)dx}{F(x_0)} \quad (7.123)
\]

or, by applying integration by parts in the numerator

\[
ES(x_0) = \frac{x_0 F(x_0) - \int_{x_0}^{x} F(x)dx}{F(x_0)} = x_0 - [F(x_0)]^{-1} \int_{-\infty}^{x_0} F(x)dx . \quad (7.124)
\]

Let us now assume that \( X \) follows a GPL distribution with known (or already estimated) parameters \( a, b \) and \( a_0, a_1, \ldots, a_r \). Then, recalling (7.13) and (7.20) we may arrive at the expression

\[
ES(x_0) = x_0 - \frac{1}{I_{F_0(x_0)}(a,b)} \int_{-\infty}^{x_0} I_{F_0(x_0)}(a,b)dx \quad (7.125)
\]

which, in the general case, can be evaluated only numerically.

If \( b \) is a positive integer, in virtue of (7.21), \( ES(x_0) \) takes on the next more appealing form

\[
ES(x_0) = x_0 - \frac{1}{d(a,b;x_0)} \sum_{k=0}^{b-1} \left( a+b-1 \right) \frac{\exp(-kP(x_0))}{\left[ 1 + \exp(-P(x_0)) \right]^{b+k+1}} \quad (7.126)
\]

where

\[
d(a,b;x_0) = \frac{1}{\left[ 1 + \exp(-P(x_0)) \right]^{b+1}} \sum_{k=0}^{b-1} \left( a+b-1 \right) \exp(-kP(x_0)) \quad (7.127)
\]

For the case of Lehmann alternative model (7.23) (i.e. \( b = 1 \)), the last formulae yield

\[
ES(x_0) = x_0 - \left[ 1 + \exp(-P(x_0)) \right] \int_{-\infty}^{x_0} \frac{1}{\left[ 1 + \exp(-P(x)) \right]^{a}} dx \quad (7.128)
\]

### 7.7 EMPIRICAL APPLICATIONS

Mittnik et al. (1998) suggested, for fitting asset returns, the use of a number of parametric distributions, and found that the asymmetric Weibull, Student-\( t \) and the asymmetric stable distributions provide the best fit according to various measures. For
a more recent application of a generalized asymmetric student-\(t\)-distribution to financial econometrics see Zhu and Galbraith (2010). Cont (2001) mentioned that the precise form of the tail of financial returns’ distribution is difficult to determine, and argued that, in order for a parametric distribution model to reproduce the properties of the empirical distribution, it must have at least four parameters (a location parameter, a scale parameter, a parameter describing the decay of the tails and an asymmetry parameter).

As far as Value-at-Risk applications in Finance are concerned, the choice of an appropriate distribution for the innovation process is an important issue, as it directly affects the ‘quality’ of the estimation of the required quantiles. Since the Normality assumption, leads to very poor performance for the resulting model, a number of alternative distributional models have been suggested including a standardized version of the skewed Student distribution, generalized gamma and the Burr distribution (see Giot and Laurent (2003), Grammig and Maurer (2000)).

In the sequel we shall illustrate the use of the GPL distribution in describing financial data. We shall proceed to statistical inference (parameter estimation) and quality assessment of the resulting probability models for four different analyses.

The first of them, is described in Section 7.7.1 and refers to the exchange rates of Euro/Canadian Dollar (CAD) for the period 1/4/1999-12/31/2011 that we used in Chapter 6 for illustrating the fitting methods and efficiency of the Polynomial Logistic distribution to describe bimodal data. Our empirical findings indicate that the GPL distribution improves significantly the quality of fitting.

The second analysis (Section 7.7.2) is performed on a data set used by Alexander et al. (2012) to elucidate the applicability of the generalized beta-generated distributions in describing the daily returns of the S&P 500 index.

In the third (Section 7.7.3) analysis we used the stock exchange price indices for 11 major capital markets and fitted appropriate GPL models on the daily log-returns of each of them for data spanning the period 1/4/1999-12/31/2012.

Finally, in the fourth analysis (Section 7.7.4), we employed the same dataset as the one used in Chapter 6 (Euro foreign exchange reference rates provided by the European Central Bank for the period 1/4/1999-12/31/2012, for major currencies);
however our analysis was performed on the daily log-returns instead of the reference rates.

Taking into account the superiority of the maximum likelihood and the least squares method over the method of moments, only those two methods were employed in our fitting procedures. In the first two analyses we used estimates derived by the maximum likelihood method, which is slightly better than the least squares estimates. However, since the calculation of the maximum likelihood estimates requires the use of time consuming iterative numerical techniques, for the last two analyses we worked with the least squares method which is much faster and provides estimates of competitive quality to the maximum likelihood estimates.

In the first two analyses (Section 7.7.1 and 7.7.2) we stress on the comparison of the quality of fitting to the respective fits provided in Chapter 6 and the already existing literature. In Section 7.7.3 we provide illustrative details for the procedure one should follow for constructing adequate distributional model for describing financial data. In both Sections 7.7.3 and 7.7.4 we investigate the potential of the fitted models to accurately approximate the values of the two most popular risk measures used in finance (Value-at-Risk and Expected Shortfall).

### 7.7.1 Distribution Fitting for Euro Foreign Exchange Reference Rates

In Chapter 6, we analyzed the Euro foreign exchange reference rates for 6 major currencies for the period 1/4/1999-12/31/2011. Exploiting Polynomial–Logistic models, we presented a detailed analysis for the Canadian Dollar (CAD) exchange rates (EUR/CAD) and provided the fitting results for the rest currencies. We shall present here the results of the analysis, by the aid of GPL models, of the EUR/CAD exchange rate data.

Table 7.4 depicts the results given in Chapter 6 when fitting polynomials of order 1 and 3 (rows with $a=1$ and $b=1$) and the corresponding results when a GLP model is used.

For the comparison of the fitted distribution models one may use several criteria such as the chi-square statistic, the log-likelihood, the Akaike Information Criterion ($AIC$) and the Schwarz or Bayesian Information Criterion ($SIC$ or $SBC$), see Burnham
and Anderson (2002). An assessment of the maximum discrepancy/distance between the empirical and the fitted cumulative distribution function values may be gained by the Kolmogorov-Smirnov statistic (KS) or the Anderson-Darling statistic (AD). It should be stressed that, the AD statistic, as compared to the KS statistic, gives appropriate weight to the tails of the distribution so that it can be used to measure goodness of tail fit.

In Table 7.4, besides the observed loglikelihood value, the Kolmogorov-Smirnov and the chi-square statistic (for the last statistic, the observed and expected frequencies were evaluated by using 10 bins), the Akaike Information Criterion, the Bayesian Information Criterion and the Anderson Darling statistic are provided. It becomes clear that, in both cases the GPL model is more effective than the respective Polynomial–Logistic with respect to all the model selection criteria.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>Chi-square</th>
<th>Loglikelihood</th>
<th>AIC</th>
<th>SBC</th>
<th>K-S</th>
<th>AD</th>
</tr>
</thead>
<tbody>
<tr>
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<td>605</td>
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<td>-5397.7</td>
<td>0.0912</td>
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<td>-5897.6</td>
<td>-5909.6</td>
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<td>6.9857</td>
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<td>3</td>
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<td>0.2</td>
<td>170</td>
<td>3035.6</td>
<td>-6059.3</td>
<td>-6071.3</td>
<td>0.0337</td>
</tr>
</tbody>
</table>

### 7.7.2 Distribution Fitting for S&P 500 Index Daily Returns

Alexander et al. (2012) exploited the generalized beta-generated distributions which were introduced by them, to create an appropriate distribution model for the daily returns of the S&P 500 index. More specifically, they considered two samples representing two very different market regimes. Sample 1, consists of the daily returns from January 2004 to December 2006, thereof covering a period where the markets were relatively stable i.e. exhibiting a low volatility. Sample 2 covers the period January 2007–December 2009, and encompasses the credit crunch and the banking crisis, when market risk factors were extremely volatile. After estimating the distribution parameters they discussed the entropy decomposition for the two samples and discussed the numerical outcomes.

In Table 7.5 we provide the values of the chi-square statistic, loglikelihood, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC),
Kolmogorov-Smirnov (K-S) and Anderson Darling (AD) statistics for the Generalized Beta-Normal model suggested by Alexander et al. (2012) (row labeled as GBN model) and Polynomial-Logistic (rows with $a=1$ and $b=1$) and GLP model of order 1 and 3.

Table 7.5. Comparing the GBN and GPL fittings for the S&P 500 index data

<table>
<thead>
<tr>
<th>$r$</th>
<th>$a$</th>
<th>$b$</th>
<th>Chi-square</th>
<th>Loglike-likelihood</th>
<th>AIC</th>
<th>SBC</th>
<th>K-S</th>
<th>AD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>1</td>
<td>15.7</td>
<td>2717.6</td>
<td>-5423.0</td>
<td>-5435.1</td>
<td>0.0406</td>
<td>2.3349</td>
</tr>
<tr>
<td>1</td>
<td>0.65</td>
<td>0.7</td>
<td>10.6</td>
<td>2716.6</td>
<td>-5425.1</td>
<td>-5433.2</td>
<td>0.0270</td>
<td>3.4326</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>9.0</td>
<td>2720.4</td>
<td>-5428.6</td>
<td>-5440.7</td>
<td>0.0360</td>
<td>2.1129</td>
</tr>
<tr>
<td>3</td>
<td>0.85</td>
<td>0.85</td>
<td>10.0</td>
<td>2717.2</td>
<td>-5422.4</td>
<td>-5434.4</td>
<td>0.0289</td>
<td>4.7204</td>
</tr>
</tbody>
</table>

Panel A. Period 2004-2006

Panel B. Period 2007-2009

Panel A offers the numerical results obtained for the period 2004-2006, while Panel B for the period 2007-2009. It is clear that the models developed for the first period are much better that the second (which is not surprising, since in the first period the volatility was low). When comparing the GBN model with the GPL models one may see that the latter beats the former in terms of the K-S statistic. However, when the AD statistic is examined, which gives appropriate weight to the tails of the distribution (and therefore it can be used to measure goodness of tail fit) the picture is mixed. The GBN model performs quite nicely for the (calm) period 2004-2006, while it does not exhibit equally good performance for the second period 2007-2009 when, due to the banking crisis, some extreme events took place. The GPL model of order 1 and order 3 seem to be capable in describing adequately the tails of the S&P 500 index data for the latter period.

7.7.3 Distribution Fitting for Stock Returns

The data analyzed in this section include stock exchange price indices for 11 major capital markets, namely the New York Stock Exchange index (NYSE), the Australian Stock Exchange Index (ASXAORD), the DAX index (DAXINDX), the CAC 40 (FRCAC40), the Financial Times All Shares (FTALLSH), the Hang Seng
(HNGKNGI), the Bombay Stock Exchange (IBOMBSE), the Swiss Market Index (SWISSMI), the Nikkei Stock Exchange Index (TOKYOSE), the Toronto Stock Exchange Index (TORONTOSE), the and the Brazilian Stock Exchange Index (TOMKBR). The data, spanning the period 1/4/1999-12/31/2012, were obtained from Datastream.

Our analysis was performed on the daily log-returns, although using simple returns will not lead to much different conclusions. We carried out our analysis on log-returns and not on the simple (geometric) returns taking into account their well substantiated, in finance literature, benefits over the simple returns.

More specifically in this section we fit appropriate Generalized Polynomial–Logistic models for each dataset and commend on their efficacy.

For illustration purposes we shall present first a detailed analysis of the daily returns of the New York Stock Exchange index (NYSE) and then provide comprehensive tables for the indices of the rest 10 markets and the 9 Euro exchange rates. As mentioned above the reference period is 1/4/1999-12/31/2012. Using the 3650 daily NYSE levels we computed the log-returns $x_1, x_2, ..., x_n$ ($n=3649$) whose histogram is depicted in Figure 7.6. The sample mean and variance was 0.0000512621 and 0.000171841 respectively (standard deviation: 0.0131088) with a minimum and maximum value $-0.102321$ and $0.0983263$ respectively.

Let us start by fitting an appropriate GPL model to our data using the regression method. The fitting will be conducted by pre-specifying the polynomial order $r$ and then carrying out an exhaustive search over $a$, $b$. More specifically, for each pair of the parameters $a$, $b$, we shall be computing the quantities (20) and then we shall fit a polynomial of order $r$ to the points $(x_i, \hat{P}(x_i))$, $i=1,2,\ldots,n$ by the method of OLS.

For the selection of the final model we may use the classical model selection criteria, namely the chi-square statistic, the log-likelihood, the Akaike Information Criterion ($AIC$) and the Schwarz or Bayesian Information Criterion ($SIC$ or $SBC$), see Burnham and Anderson (2002). Moreover an assessment of the maximum discrepancy/distance between the empirical and the fitted cumulative distribution function values may be gained by the Kolmogorov-Smirnov statistic ($KS$) or the Anderson-Darling statistic ($AD$).
Figure 7.6. Histogram of daily NYSE stock returns for the period is 1/4/1999-12/31/2012

Table 7.6 displays the values of all aforementioned quantities for \( r=3 \) (i.e. we have fitted GPL distributions associated with polynomials of order 3 - we shall briefly refer to them as GPL distributions of order 3) and several choices of the parameters \( a \) and \( b \).

The best model, according to the chi-square values corresponds to \( a=0.25 \), \( b=0.30 \) and the associated fitted polynomial is as follows

\[
P(x) = -0.7914 + 420.7051 x + 343.1141 x^2 + 93.2777 x^3
\]

(7.129)

or by rounding its coefficients to the nearest integer value

\[
P(x) = -1 + 421 x + 343 x^2 + 93 x^3.
\]

(7.130)

It is noteworthy that when the values of the parameters \( a \) and \( b \) are close to each other, the tails of the resulting distribution are very similar. This is in agreement to the empirical finding reported by Jondeau and Rockinger (2003), that for actual returns, the left and right tails behave very similarly.

Apparently the choice of optimal values of the parameters \( a \), \( b \) will not be the same should the selection criterion be different, e.g. minimum value of \( KS \) or \( AD \) statistics. However, the performance of the resulting model will not be substantially different than the one created by couching on the chi-square statistic.

---

\(^{1}\) For the evaluation of the chi-square statistic we used 10 bins.
Figure 7.7 depicts the histogram of the NYSE log-returns along with the fitted GPL probability density function corresponding to the polynomial (7.129) and the probability density function of the normal distribution with mean 0.0000512621 and variance 0.000171841 (as estimated from the raw data). Apparently the GPL model offers substantially better fitting than the normal model, both in the peakness and the tails.

It is noteworthy that, should one use the maximum likelihood method (by minimizing (7.90) with respect to $a$, $b$ and the 4 polynomial coefficients or equivalently by solving the system of equations (7.91), (7.92), (7.93)) would end up with a GPL distribution with

$$a = 52.5, \quad b = 34.3, \quad P(x) = 0.422522 + 17.5093x + 39.7501x^2 + 30.081x^3 \quad (7.131)$$
and much higher chi-square KS and AD values than the ones achieved by carrying out exhaustive search on \( a \) and \( b \) and subsequently fitting the polynomial \( P(x) \) by the regression method.

**Figure 7.7.** Histogram of daily NYSE stock returns with fitted GPL (sharper curve) and normal distribution

![Histogram of daily NYSE stock returns](image)

The quality of fitting might be considerably improved by fitting to our data (by the regression method) a GPL distribution of order 5. The best model deduced by the regression method is the one with \( a = 0.4 \), \( b = 0.6 \) and

\[
P(x) = -0.9 + 297.2x - 1000.6x^2 - 45532.1x^3 + 190342.8x^4 + 4419066.1x^5 \tag{7.132}
\]

The respective log-likelihood value increases to 11184.9 with the chi-square statistic shrinking to 21.4 (note that this value is smaller than the chi-square critical value for 9 degrees of freedom, which equals 21.7).

It is worth mentioning that, in some cases we may be able to spot a polynomial of higher order that is not increasing in the whole real axis and exhibits better performance than the fitted polynomial of the same order obtained under the monotonicity condition. Should such a polynomial be increasing in the area where our data spread out, we can effectively use it as a generator of an associated truncated distribution (restricting our parametric model’s support to the data’s range).
For comparison reasons, we have also fitted the NYSE data to the following parametric models, members of the GPL family of distributions:

i. a GPL distribution of order $r = 3$ with $a = 1$, $b = 1$ (this special case is in fact a Polynomial-Logistic distribution of order 3, as introduced and studied in Chapter 6,

ii. a GPL distribution of order $r = 1$ and

iii. a classical Logistic distribution (i.e. a GPL distribution of order $r = 1$ with $a = 1, b = 1$)

The fitted distributions for each of the aforementioned cases were as follows

i. $a = 1, b = 1$ and $P(x) = -0.0651726 + 140.619x + 390.851x^2 + 362.125x^3$,

ii. $a = 0.4, b = 0.5$ and $P(x) = -0.602944 + 268.5806x$,

iii. $a = 1, b = 1$ and $P(x) = 0.00062192 + 138.024x$.

Note again that the optimal choices of the parameters $a$ and $b$, with the exception of model 3 which is not statistically acceptable, pertain to values that are close to each other, thereof obtaining distributional models with similar left and right tails (c.f. Jondeau and Rockinger (2003)).

In Table 7.7 we provide the values of the most commonly used goodness of fit criteria for all models fitted so far to the NYSE data. The models’ labeling is as follows: 1-4 refer to the models given in (7.129), (7.130), (7.131), (7.132) respectively while 5-7 refer to the models described in (i)-(iii) above.

<table>
<thead>
<tr>
<th>Model</th>
<th>$r$</th>
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<th>Loglikelihood</th>
<th>AIC</th>
<th>SBC</th>
<th>KS</th>
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</thead>
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</table>

It is clear that the GPL model of order 5 outperforms all other models displayed above in terms of any of the goodness of fit criteria.

As mentioned in Section 6, two risk measures widely used in financial econometrics are the Value at Risk and the Expected Shortfall. Let us next investigate
how accurate are the models 1-7 in terms of providing appropriate approximations for these risk benchmarks. In Table 7.8 we present, for several values of $p$ ($0.01 \leq p \leq 0.10$) the empirical values of $VaR_p$ for the daily NYSE stock returns (calculated directly from the available data as their $p$-th quantile) and the approximate values generated by each of the 7 fitted parametric models.

The formula used for the computation of $VaR_p$ for each model is the one shown in (7.118). For comparison reasons we have added a column (labeled as column 8) where we report the values of $VaR_p$ deduced by the aid of the corresponding classical parametric model i.e. by assuming that the log-returns follow a normal distribution with the mean and variance estimated from the available data.

In order to grasp a clearer view of the quality of each model, we computed in Table 7.9 the absolute differences between the empirical values of $VaR_p$ and the approximate values generated by each of the 8 fitted models.

The last row of the table displays the mean of the absolute differences for each model, a quantity that can be used as an overall measure of the capacity of the model to capture the Value at Risk of the data under scrutiny. It is clear that the best model is by far the one established by the GPL distribution of order 5 (model 4), while the classical Logistic distribution (model 7) and the classical parametric model (model 8) exhibits quite poor performance.

Finally, let us check how do the models perform when used to capture the Expected Shortfall ($ES$) of the data. To illustrate that, we have evaluated for each of the 8 models, the $ES$ at the points indicated by the $VaR_p$ values. The approximate (absolute) values for $ES$ are displayed in Table 7.10 while Table 7.11 depicts the absolute differences between the empirical (absolute) values of $ES$ (shown in the second column of the table) and the approximate values obtained by practicing each of the 8 fitted models. The last row of the table displays the mean of the absolute differences for each model and clearly indicates the superiority of the alternatives arising from the GPL family against the classical parametric model (model 8). Needless to say, the best model is by far the one established by the GPL distribution of order 5 (model 4).
### Table 7.8. Empirical and approximate values of \( \text{VaR}_p \) for the daily NYSE stock returns

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### Table 7.9. Absolute differences between empirical and approximate values of \( \text{VaR}_p \) for the daily NYSE stock returns

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Table 7.11. Absolute differences between empirical and approximate values of Expected Shortfall for the daily NYSE stock returns

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Mean | **0.00171744** | **0.00174982** | **0.00670837** | **0.00034456** | **0.00272104** | **0.00217588** | **0.00459095** | **0.00672201** |
The analysis detailed above, was also applied for the Stock Exchange log-returns of the Rest of 10 major international markets mentioned at the beginning of this section, that is the Australian Stock Exchange Index (ASXAORD), the DAX index (DAXINDEX), the CAC 40 (FRCAC40), the Financial Times All Shares (FTALLSH), the Hang Seng (HNGKNGI), the Bombay Stock Exchange (IBOMBSE), the Swiss Market Index (SWISSMI), the Nikkei (TOKYOSE), the Toronto Stock Exchange

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Table 7.12. Optimal choices of the parameters a and b for fitting the daily log-returns of major international markets
Index (TORONTOSE) and the Brazilian Stock Exchange Index (TOTMKBR). The results are summarized in Table 7.12 where the values of the most commonly used goodness of fit criteria are displayed when practicing GPL distributions of order \( r = 1, 3, 5 \).

The suggested parameters \( a \) and \( b \) were derived by carrying out an exhaustive search and taking into account not only the value of the chi-square statistic (which would apparently be affected if the number of bins used was changed) but the values of the KS and AD statistics as well. Taking into account the fact that, as Jondeau and Rockinger (2003) have stated, the tails of the returns distributions are very similar, one may speed up the search procedure by restricting to \( a \) and \( b \) values that are close to each other.

To facilitate the practitioner who might wish to use the suggested parametric models for an analysis of the respective stock exchange data, we provide, in Table 7.13, the respective fitted polynomial for each case (with its coefficients rounded to 1 decimal point).

It is noteworthy that, in most cases, the values of all the statistics which are traditionally exploited as optimality criteria for model selection (chi-square, likelihood, AIC, SBC, KS, AD) improve when a higher order polynomial is used. In some cases the chi-square statistic may exhibit a puzzling picture when compared to the rest criteria (see e.g. HNGKNGI or TOTMKBR), however this might be alleviated if a different number of bins was used for it (in all reported values, 10 bins have been used).

It should also be emphasized that all fitted polynomials of order 1 and 3 where calculated by minimizing the OLS statistic (sum of squares of error terms) under appropriate necessary and sufficient conditions on the unknown parameters (polynomial coefficients) which guarantee that the resulting polynomial is increasing. However this is not feasible for a polynomial of order 5, since one cannot present an explicit set of necessary and sufficient conditions on the polynomial parameters securing the monotonicity of a polynomial of order 5; to overcome this, we initially carried out an unconstrained minimization search and then checked whether the resulting polynomial is increasing in the range of the data used. If this did not hold true, we proceeded to a constrained minimization using necessary (and not sufficient)
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conditions for the polynomial to be increasing. As a consequence, in these cases, there
is room for further improvement of the fitted distribution, since the search of the
polynomial was carried out on a subspace of the feasible set (all increasing
polynomials of order 5).

The fitted models described in Table 7.12 and Table 7.13 apparently could be
used for estimating the Value at Risk and Expected Shortfall when one considers
investment holdings in any of the 10 major international markets we studied. In Table
7.14 we present the VaR and (absolute) ES at levels 1% and 5% values obtained by
the models reported in Table 7.12 and Table 7.13.

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For comparison reasons, we have included in the last 2 rows of each block the corresponding empirical values (obtained from the raw data) as well as the values offered by the classical parametric model. Note that all tabulated values have been multiplied by $10^8$.

A careful inspection of Table 7.14, readily reveals that the VaR and ES estimates deduced by the GPL models are quite accurate and beat the ones obtained by the classical parametric model.

### 7.7.4 Distribution Fitting for Euro Foreign Exchange Returns

The applicability of the GPL models is further supported by their remarkable adaptability to a variety of financial data, besides the ones already analyzed.

The data analyzed in this section include the Euro foreign exchange reference rates provided by the European Central Bank. We have used the Euro exchange rates for 9 major currencies, namely the Australian Dollar (AUD), Brazilian Real (BRL), Canadian Dollar (CAD), Swiss Franc (CHF), UK Pound (GBP), Hong Kong Dollar (HKD), Indian Rupee (INR), Japanese Yen (JPY) and US Dollar (USD). The data used, refer to the period 1/4/1999-12/31/2012.

In Tables Table 7.15, Table 7.16, and Table 7.17, which are similar in structure to Tables Table 7.12, Table 7.13 and Table 7.14, we summarize the results of fitting the GPL distribution to the Euro foreign exchange reference rates for 9 major currencies (Australian Dollar (AUD), Brazil Real (BRL), Canadian Dollar (CAD), Swiss Franc (CHF), UK Pound (GBP), Hong Kong Dollar (HKD), Indian Rupee (INR), Japanese Yen (JPY) and US Dollar (USD)) for the period 1/4/1999-12/31/2012. In contrast to the analysis performed in Chapter 6, we are now working not on the reference rates values but on the daily log-returns of them.

The remarks made earlier for the case of Stock Exchange log-returns, are also valid for the fitting of the Euro foreign exchange reference rates. Table 7.15 shows that, in most cases, the values of all the traditional model selection criteria/statistics improve when a higher order polynomial is used. In addition, a careful inspection of Table 7.17, shows that the VaR and ES estimates deduced by the GPL models are quite accurate and beat the ones deduced by the classical parametric model.
### Table 7.15. Optimal choices of the parameters \( a \) and \( b \) for fitting the Euro foreign exchange logreturns for 9 major currencies

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### Table 7.16. Fitted polynomial for the optimal choices of the parameters \( a \) and \( b \) shown Table 7.15

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<th>( r )</th>
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<th>( b )</th>
<th>Fitted polynomial</th>
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<td>0.27</td>
<td>0.6 + 738.2 ( x ) + 1321.8 ( x^2 ) + 789. ( x^3 )</td>
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<td>0.7</td>
<td>0.5</td>
<td>0.7 + 432.6 ( x ) + 1470.9 ( x^2 ) - 114494.8 ( x^3 ) - 199410.3 ( x^4 ) + 15739113.9 ( x^5 )</td>
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<td>( b )</td>
<td>( \text{VaR}_{0.01}(\times 10^8) )</td>
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<td>0.9</td>
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<td>0.9</td>
<td>( 0.3 + 372.8 x + 994.5 x^2 - 313249.9 x^3 - 1859804.6 x^4 + 226225669.3 x^5 )</td>
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<td>( -0.02 + 548.4 x + 1787.3 x^2 + 2038. x^3 )</td>
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<td>0.8</td>
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Table 7.17. Comparison of VaR and ES values obtained by the models reported in Table 7.15 and Table 7.16
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7.8 CONCLUDING REMARKS

It is widely known that the empirical distribution of many financial assets' returns (e.g. stocks, exchange rates etc.) is usually highly peaked and displays heavy tails, which by far exceed what could be accounted for by a Gaussian (Normal) distribution. In many cases the logistic distribution has been found valuable in modeling such data. However, the symmetry of the logistic distribution in left and right tails (losses and gains) often seems violated in practice. Hence generalizations that allow asymmetry would be valuable in empirical modeling and forecasting.

In the present chapter we have introduced a multi-parameter family of distributions which nests the classical logistic distribution and is capable of modeling asymmetry and heavy tail phenomena. Several estimation techniques are discussed for the parameters of the distribution and formulae are developed for its moments, absolute moments and important quantities related to financial risk (Value-at-Risk and Expected Shortfall).

A detailed illustration is given how this class of distributions can be exploited to establish appropriate parametric model for the New York Stock Exchange index (NYSE) daily log-returns. In addition we present comprehensive results for the quality of fitting achieved when analyzing the daily returns of eleven major international markets and the Euro foreign exchange daily returns for nine major currencies.
CHAPTER 8. CONCLUSIONS AND SUGGESTIONS

8.1. A BRIEF SUMMARY OF THE CONTRIBUTION OF THE THESIS

In the present thesis we have dealt with several problems of the financial risk management area pertaining to bank supervision and risk modeling.

Securing the appropriate modeling of the risks of a bank run is of crucial interest, so, in a well maintained financial system, the state has to establish mechanisms responsible to promote the safety and soundness of depository institutions, including banks, by appropriately identifying, monitoring, and addressing risks. For the supervisory monitoring of banks, some quantitative methods applied on useful financial ratios (risk factors) that are often in use to measure the overall soundness of a bank’s operation, have to be practiced. A useful tool for accomplishing the supervisory task in this framework is the family of the CAMEL risk factors which has been extensively exploited in the financial risk management area.

Motivated by the above facts, our aim was to contribute to the following three issues, which are of major importance both for the regulator (who is responsible for monitoring the financial conditions of commercial banks, enforcing related legislation and regulatory policy) and the institutions being supervised (which are usually required to develop their own internal models as a basis for measuring their market risk capital requirements, subject to strict quantitative and qualitative standards):

a. Identifying which risk factors contain useful information for bank failure, and how quick they deteriorate over time.

b. Investigating how one could develop an early warning system that will provide timely signs of an on-coming crisis.

c. Introducing new distributions capable to model effectively the stochastic behavior of loss elicited by risky investments.

Our contribution on the aforementioned topics (a) and (b) involves, among others
(i) a suggestion to use a monitoring technique based on mobility indices in order to track developing trends of the bank system (improvements vs. deteriorations of capitalization),

(ii) the documentation of three stages in the bank life cycle (prior to failure), which are determined according to the groups of CAMEL factors that send out signals for bank failure.

To achieve (i), we used a panel dataset whose cross section consists of all US commercial banks and its time series dimension of the period 1992-2009 (annual intervals), and set-up a Markov Chain whose states are associated with the four buckets generated by exploiting the FDIC discretized version of Capital Ratio. Then we estimated the transition probabilities between capitalization buckets; including an absorption state that described the bank default. The benefit of this approach is that it allows the quantification of migration probabilities of banks across the capitalization spectrum, among which those related to the default state are of special interest. Taking advantage of the large period spanned by our dataset, we proceeded to a comparison of the bank system mobility, between the 90's and the 00's. It is of great importance that the period under study covers a time horizon in which the Basel requirements are in place.

Our empirical analysis documented several substantial differences in the anatomy of the two periods: system mobility has shown a step increase in the 00's, and what is more important, mobility takes the form of increased probability of capitalization deterioration.

Our approach, besides the comparison of different financial periods, offers a methodology that could also be practiced in the future for highlighting possible developing trends, thereof getting an early warning of on-coming systemic crisis in the bank sector. Such a system will be useful both for market participants, but especially for the regulator.

To achieve the objective (ii), we employed a panel dataset whose cross section consists of all US commercial banks insured by the FDIC and its time series dimension of the period 2000:q1-2011:q4 (quarter intervals).

We justified that, as already mentioned in the relevant literature, there are no substantial differences between (subsequently) failed banks and their non-failed peers.
up until about 20 quarters prior to failure. Focusing on the period starting from that point till the bank failure, we used ex ante information on several key CAMEL factors, to explore how many quarters prior to failure, they may contain useful signals for the excess vulnerability of banks. The novelty of the empirical analysis was that it focuses on signals pertaining not to specific banks, but rather to groups of banks exposed to certain level of the risk factors.

Our empirical findings indicated that, four to five years prior to failure three of the CAMEL risk factors contain significant forecasting power for future failures. Looking at a 3 year horizon before failure, we uncovered that the set of significant predictors is doubled in size (three additional CAMEL factors jump in). Finally, two years prior to failure, most of the risk factors (10 out of 11) contain significant forecasting power. Combining the findings of several alternative statistical techniques (namely, analysis of contingency tables, logistic regression, Discriminant Analysis and Correspondence Analysis) we substantiated that, according to the group of CAMEL factors that send out signals for bank failure, one may break up the bank life cycle (prior to failure) to 3 stages: 3-5 years prior to failure (Stage I), 3 years prior to failure (Stage II) and finally 1 year prior to failure (Stage III). Finally, a single index for failure forecasting in each of the three stages of bank failure lifecycle was developed by practicing techniques from the area of Principal Component Analysis.

The reported results on the CAMEL factor ex-ante forecasting capability, are particularly useful for the regulator, when the need arises to quantify the excess vulnerability of specific groups of banks across different time horizons (with respect to predicting bank default).

Our contribution on topic (c) includes the introduction and study of two new parametric distribution models capable of describing efficiently several financial assets. Their significance in practical applications can be easily substantiated by the fact that the normality assumption for various asset returns is not supported by empirical evidence; one may spot out numerous publications in the financial literature, which verify that the distributions of several assets (e.g. stock returns, exchange rates etc.) are usually quite highly peaked and heavy tailed as compared with normal distributions. Therefore, alternative models with heavier tails will highly facilitate the process of establishing a better fit to the data of interest.
The genesis of our first model was inspired by the well-known property of the Logistic distribution that the logit transformation \( \log(F(x))/(1-F(x)) \) of its cumulative distribution function \( F(x) \) is a linear function of \( x \). The new model, named Polynomial-Logistic distribution, was created by considering a family of distributions having respective logit transformation of Polynomial type (instead of a linear). An important feature of the new family of distributions, is that it accommodates a variety of models with a plethora of shapes (unimodal, bimodal, symmetric and non–symmetric), a fact making it quite powerful in fitting real data. Besides the theoretical results on the several characteristics of the Polynomial-Logistic family of distributions, we give an illustration on how it can be used to gain an accurate parametric model describing the Euro foreign exchange reference rates of 6 major currencies (Canadian Dollar, US Dollar, Australian Dollar, Swiss Franc, Japanese Yen, and UK Pound) for the period 1/4/1999-12/31/2011. The quality of fitting in these data is very promising.

In the same spirit, a further generalization of this model was introduced by adding two parameters that can be used to control for the asymmetry and the decay of tails. The new parametric family, named Generalized Polynomial-Logistic distribution, nests a variety of distribution models (unimodal, bimodal, symmetric and non–symmetric) and includes as special cases the generalized logistic distribution which has been extensively studied in the literature, as well as the Polynomial-Logistic distribution mentioned above. The new family of distributions can be used in several frameworks, including modeling of the extreme minima and maxima daily returns, developing appropriate models for describing extremes in environmental studies, as models for income distributions etc.

Several properties related to the generation of random variables following the Generalized Polynomial-Logistic distribution, aging properties, as well as exact expressions for its moments and absolute moments have been studied in detail. We also presented methods for estimating its parameters, along with formulae for evaluating the Value at Risk and Expected Shortfall metrics. Finally, we exploited the family of Generalized Polynomial-Logistic distributions to model the daily log-returns of eleven major international capital markets (United States, Canada, England, Switzerland, Germany, France, Hong Kong, Japan, India, Australia and Brazil) for the period 1/4/1999-12/31/2012 and the Euro foreign exchange daily returns for 9 major
currencies (Canadian Dollar, US Dollar, Australian Dollar, Swiss Franc, Japanese Yen, UK Pound, Hong Kong Dollar, Indian Rupee and Brazil Real) for the same period. The quality of fitting in these data seems to be excellent and surpasses the fit provided by the use of the Normal distribution, the classical Logistic distribution and the Polynomial Logistic distribution.

8.2. SUGGESTIONS FOR FUTURE RESEARCH

In the present section we shall present some ideas for further research on the topics covered in our thesis.

In Chapter 4 we conducted a comparison of the two past decades in terms of US banks’ capitalization mobility and persistence. To achieve that, we adopted a Markov Chain setup, based on the FDIC discretized version of Capital Ratio. The underlying assumption being made in our analysis is that, the migrations among the FDIC buckets exhibit a first order Markov dependence, therefore only the first order transition probabilities were estimated and then used in the formulas of mobility indices and distance metrics.

Future research in this area could explore possible higher order time dependence in the migrations among the buckets. Additionally, one could allow for geographic and size dependent transition probabilities. Then a higher Markov Chain could be established (for the first case) as well as on inhomogeneous Markov Chain (for the second one) and adjust accordingly the classical mobility indices and distance metrics so as a comparison of the bank system mobility between several time periods be carried out and an early warning of deterioration trends in the bank sector be built.

Apparently, due to the stronger assumption imposed on the dependency of the transition probabilities (which is more realistic than the first order Markov dependence) and the provision for geographic and size discrimination, the new models are expected to improve the quality of statistical inference and the reliability of the conclusions derived from it.

In Chapter 5 we practiced several statistical methods in order to investigate which bank metrics, based on the well documented collection of CAMEL factors, exhibit ex ante long term forecasting power for bank failures. The novelty of the
empirical analysis carried out there, was that it focused on signals pertaining not to specific banks, but rather to groups of banks exposed to certain level of the risk factors. The main statistical tools employed in our analysis were logistic regression models and several multivariate techniques. Davis and Karim (2008a), (2008b), compared several statistical and intelligence techniques that one may use for building up early warning systems for banking crises. Their empirical analysis indicated that the choice of the estimation models makes a big difference in terms of indicator performance and crisis prediction. They argued that statistical techniques, especially logistic regression (logit models), performs better as a global early warning system while an intelligence-based model, such us Signal Extraction Early Warning System, is preferable as a country-specific one. Moreover, they compared the logit and binomial tree approaches with regards to predicting the 2007 subprime crisis in the US. Finally, West (1985) combined the Logit model with factor analysis, to measure and describe banks’ financial and operating characteristics and he received important descriptive variables similar to those used for CAMELS ratings. Motivated by the above findings, we may suggest as a possible issue of future research a detailed comparison of the logistic approach used in the present thesis to models using intelligence techniques, classification trees and Factor Analysis models; needless to say, the suggested comparison will have to parallel our approach, that is it will be performed on models analyzing signals pertaining not to specific banks, but to groups of banks exposed to certain level of the risk factors.

In Chapters 6 and 7, we introduced new parametric distribution families which offer quite remarkable adaptability in real data arising in finance. Taking into account that, despite the extensive research work on the field of modeling financial assets, it seems that there is a continuous need to enrich the distribution gallery with more sophisticated and/or simple to implement models, one might consider further generalizations and/or refinements of the suggested models. For example, a straightforward but still promising extension of the generalized Polynomial-Logistic distribution studied in Chapter 7 may be achieved by considering a cumulative distribution function of the form (see Alexander et al. (2012))

\[
F(x; a, b; c) = \frac{1}{B(a, b)} \int_0^{\frac{\beta}{\alpha}} \frac{\beta}{\beta - 1} (1 - t)^{\beta - 1} \, dt = \frac{B_{\beta}(a, b)}{B(a, b)} = I_{-\beta}(a, b) \tag{8.1}
\]
where $a, b, c$ are positive shape parameters and $F_0(x)$ is a baseline cumulative distribution function that is associated to the Polynomial-Logistic distribution, i.e.

$$F_0(x) = \frac{1}{1 + \exp(-P(x))}, \quad -\infty < x < \infty. \quad (8.2)$$

A simpler model may be created by considering the special case $a = 1$, thereof obtaining the following cumulative distribution function

$$F(x,a,b;c) = \frac{1}{B(1,b)} \int_0^{F_0(x)} (1-t)^{b-1} \, dt = 1 - (1 - F_0(x))^{b-1}, \quad -\infty < x < \infty. \quad (8.3)$$

The last model is much simpler than the generalized Polynomial-Logistic model, however the use of the extra parameter $c$, might be sufficient for getting nice fittings for financial data. We are planning to further elucidate on this in our future research work.

Another direction for further research in the distribution modelling would be to explore how the new models perform to other types of data. For example, as the related actuarial literature reports, asymmetry and heavy tail phenomena are quite likely to occur when analyzing claim sizes raised by the customers of an insurance company. Moreover describing data related to extremes in environmental studies as well as catastrophic events (e.g. tornados or earthquakes) will probably offer nice fields of application of the models introduced in this thesis. Finally, one more area that may probably benefit a lot from the use of the new models studied in the present thesis is the rapidly developing area of big data analytics where a lot of research activity has been noticed in the last few years. Most of the classical distributions miserably fail to describe satisfactorily the big datasets that the data-mining analysts have at hand today, so the flexibility offered by our models, may be proved quite beneficial in the task of describing accurately the underlying structures.

As a final remark we mention that the new distributions introduced in the present thesis might also be used as a parametric model for the innovation process in ARCH, GARCH and stochastic volatility models for fitting asset returns. This topic will be further explored in our future works.
REFERENCES


Basel Committee on Banking Supervision (BCBS) Bank for International Settlements (BIS): www.bis.org/bcbs

Basel Committee on Banking Supervision (BCBS), Charter,(Jan 2013) BIS:

Basel I: (July 1988), International convergence of capital measurement and capital standards, Bank for International Settlements (BIS)

Basel II: (June 2006), International convergence of capital measurement and capital standards, Revised Framework, Bank for International Settlements (BIS)

Basel III: (June 2011), A global regulatory framework for more resilient banks and banking systems December 2010 (rev June 2011), Bank for International Settlements (BIS)

Bell, T.B., (1997), Neural nets or the logit model? A comparison of each model’s ability to predict commercial bank failures., International Journal of Intelligent Systems in Accounting, Finance and Management 6, 249–64.


Concordat, (1975), Report to the governors on the supervision of banks; foreign establishments, Bank for International Settlements (BIS).


Federal Deposit Insurance Corporation (FDIC), (2014), webpage: www.fdic.gov


Kahn C.M., Santos, J.A.C., (2001), Allocating bank regulatory powers: Lender of last resort, deposit Insurance and Supervision


Koutras, V. M., Drakos, K., (2013), A migration approach for USA banks’capitalization: Are the 00s the same with the 90s? International Review of Financial Analysis 30, 131-140.


Li, B. B. and De Moor, B., (1997), A simple approximation relation between the symmetric generalized logistic and the student’s t distributions, Communications in Statistics- Simulation and Computation 26, 1029-1039.


Pfarr, C., Schmid, A. and U. Schneider, (2010), REGOPROB2: Stata module to estimate random effects generalized ordered probit models (update), Statistical Software Components, Boston College Department of Economics.

Piyu, Y., (1992), Data envelopment analysis and commercial bank performance: A Primer with Applications to Missouri Banks, IC2, Institute, University of Texas at Austin.


Theil, H., (1972), Statistical Decomposition Analysis, North Holland, Amsterdam.


Yang W.K, Chun E.S., Duncan M.C., Failure Resolution Methods, Costs, and Policy Implications: Experiences in the U.S.and Korea, FDIC.


APPENDIX

CHAPTER 4.

Figure 4.3  Number of defaults per year in USA for the period 1993-2008

Figure 4.4  Percentage of defaults per year in USA for the period 1993-2008
Figure 4.6 Percentage of USA banks belonging to the Adequately Capitalized category per year for the period 1992-2009

Figure 4.7 Percentage of USA banks belonging to the Undercapitalized category per year for the period 1992-2009
**Figure 4.8** Percentage of USA banks belonging to the Significantly Undercapitalized category per year for the period 1992-2009

**Figure 4.9** Percentage of USA banks belonging to the Critically Undercapitalized category per year for the period 1992-2009
Figure 4.15  Transition probabilities (x100) to default for USA banks belonging to each capitalization category

Notes: On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively.

Figure 4.16  Transition probabilities (x100) to the Critically Undercapitalized category for USA banks belonging to each capitalization category

Notes: On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively.
Figure 4.17 Transition probabilities (x100) to the Significantly Undercapitalized category for USA banks belonging to each capitalization category

Notes: On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively.

Figure 4.18 Transition probabilities (x100) to the Undercapitalized category for USA banks belonging to each capitalization category

Notes: On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively.
**Figure 4.19** Transition probabilities (x100) to the Adequately Capitalized category for USA banks belonging to each capitalization category

![Graph showing transition probabilities between capitalization categories from 1992-2000 and 2001-2009.]

**Notes:** On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively.

**Figure 4.20** Transition probabilities (x100) to the Well Capitalized category for USA banks belonging to each capitalization category

![Graph showing transition probabilities between capitalization categories from 1992-2000 and 2001-2009.]

**Notes:** On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively.
Figure 4.24 Transition probabilities (x100) to the Critically Undercapitalized category for USA banks belonging to each capitalization category

Notes: On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively

Figure 4.25 Transition probabilities (x100) to the Significantly Undercapitalized category for USA banks belonging to each capitalization category

Notes: On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively
**Figure 4.26** Transition probabilities (x100) to the Undercapitalized category for USA banks belonging to each capitalization category

**Notes:** On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively

**Figure 4.27** Transition probabilities (x100) to the Adequately Capitalized category for USA banks belonging to each capitalization category

**Notes:** On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively
Figure 4.28 Transition probabilities (x100) to the Well Capitalized category for USA banks belonging to each capitalization category

Notes: On the horizontal axis 1, 2, 3, 4 and 5 denote Critically Undercapitalized, Significantly Undercapitalized, Undercapitalized, Adequately Capitalized, and Well Capitalized categories respectively.
CHAPTER 5.

Figure 5.6. Graphs of roa for non-failed and failed banks up to 42 quarters before failure

Notes: On the horizontal axis -40, -30, -20 and -10 denote quarters prior to failure.

Figure 5.7. Graph of cov for non-failed (smooth line) and failed (dotted line) banks up to 42 quarters before failure

Notes: On the horizontal axis -40, -30, -20 and -10 denote quarters prior to failure.
**Figure 5.8.** Graph of nim for non-failed (smooth line) and failed (dotted line) banks up to 42 quarters before failure

![Graph of nim](image)

*Notes:* On the horizontal axis -40, -30, -20 and -10 denote quarters prior to failure.

**Figure 5.9.** Graph of eff for non-failed (smooth line) and failed (dotted line) banks up to 42 quarters before failure

![Graph of eff](image)

*Notes:* On the horizontal axis -40, -30, -20 and -10 denote quarters prior to failure.
**Figure 5.10a.** Comparison of defaults per exposure class of Camel factors for 5 year horizon

![5 years horizon graph]

**Figure 5.10b.** Comparison of defaults per exposure class of Camel factors for 4 year horizon

![4 years horizon graph]
Figure 5.10c. Comparison of defaults per exposure class of Camel factors for 3 year horizon

![3 years horizon](image)

Figure 5.10d. Comparison of defaults per exposure class of Camel factors for 2 year horizon

![2 years horizon](image)
Figure 5.10e. Comparison of defaults per exposure class of Camel factors for 1 year horizon

Figure 5.11a. Comparison of defaults per exposure class for rbc Camel factor, across the 5 horizons
Figure 5.11b. Comparison of defaults per exposure class for nd Camel factor, across the 5 horizons

Figure 5.11c. Comparison of defaults per exposure class for equ Camel factor, across the 5 horizons
Figure 5.11d. Comparison of defaults per exposure class for roa Camel factor, across the 5 horizons

![CAMEL factor: roa](image)

Figure 5.11d. Comparison of defaults per exposure class for cov Camel factor, across the 5 horizons

![CAMEL factor: cov](image)
**Figure 5.11d.** Comparison of defaults per exposure class for nim Camel factor, across the 5 horizons

**Figure 5.11e.** Comparison of defaults per exposure class for eff Camel factor, across the 5 horizons
Figure 5.11f. Comparison of defaults per exposure class for past30 Camel factor, across the 5 horizons

Figure 5.11f. Comparison of defaults per exposure class for nco Camel factor, across the 5 horizons
Figure 5.11g. Comparison of defaults per exposure class for rwa Camel factor, across the 5 horizons

![CAMEL factor: rwa](image)

Figure 5.11h. Comparison of defaults per exposure class for rwa Camel factor, across the 5 horizons

![CAMEL factor: lgr](image)
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**Table 5.8.** Correspondence analysis coordinates and quality metrics for a 4 year horizon
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| 11 | 0.017 | 0.592 | 0.064 | 0.324 | 0.008 | 0.002 | 3.388 | 0.549 | 0.193 | 1.125 | 0.035 | 0.021 |
| 12 | 0.023 | 0.238 | 0.015 | 0.155 | 0.011 | 0.001 | 0.754 | 0.153 | 0.013 | 0.692 | 0.074 | 0.111 |
| 13 | 0.025 | 0.374 | 0.010 | -0.037 | 0.001 | 0.000 | -0.929 | 0.372 | 0.021 | -0.072 | 0.001 | 0.000 |
| 14 | 0.026 | 0.532 | 0.043 | -0.306 | 0.017 | 0.002 | -1.932 | 0.416 | 0.099 | -1.245 | 0.099 | 0.041 |

ncl

| 21 | 0.016 | 0.759 | 0.019 | -1.093 | 0.283 | 0.019 | 1.084 | 0.171 | 0.018 | -1.913 | 0.305 | 0.057 |
| 22 | 0.024 | 0.789 | 0.008 | -0.897 | 0.750 | 0.019 | -0.195 | 0.022 | 0.001 | -0.227 | 0.017 | 0.001 |
| 23 | 0.026 | 0.770 | 0.003 | -0.113 | 0.032 | 0.000 | -0.497 | 0.380 | 0.006 | 0.638 | 0.359 | 0.010 |
| 24 | 0.026 | 0.826 | 0.025 | 1.583 | 0.769 | 0.065 | 0.017 | 0.000 | 0.000 | 0.724 | 0.057 | 0.014 |

equ

| 31 | 0.017 | 0.590 | 0.043 | 0.206 | 0.005 | 0.001 | 2.681 | 0.524 | 0.125 | 1.212 | 0.061 | 0.026 |
| 32 | 0.024 | 0.216 | 0.010 | 0.050 | 0.002 | 0.000 | 0.406 | 0.070 | 0.004 | 0.772 | 0.145 | 0.014 |
| 33 | 0.025 | 0.549 | 0.009 | -0.123 | 0.013 | 0.000 | -1.019 | 0.528 | 0.026 | -0.172 | 0.009 | 0.001 |
| 34 | 0.024 | 0.542 | 0.022 | -0.067 | 0.001 | 0.000 | -1.243 | 0.307 | 0.038 | -1.436 | 0.234 | 0.050 |

rse

| 41 | 0.022 | 0.430 | 0.038 | -1.419 | 0.347 | 0.044 | 0.321 | 0.011 | 0.002 | 1.085 | 0.072 | 0.026 |
| 42 | 0.024 | 0.623 | 0.013 | -0.937 | 0.463 | 0.021 | -0.284 | 0.026 | 0.002 | 0.851 | 0.134 | 0.017 |
| 43 | 0.024 | 0.048 | 0.012 | 0.082 | 0.004 | 0.000 | -0.293 | 0.031 | 0.002 | -0.248 | 0.013 | 0.001 |
| 44 | 0.021 | 0.696 | 0.065 | 2.479 | 0.578 | 0.129 | 0.323 | 0.006 | 0.002 | -1.838 | 0.112 | 0.071 |

cov

| 51 | 0.022 | 0.586 | 0.070 | -1.890 | 0.340 | 0.080 | 1.516 | 0.134 | 0.052 | -1.829 | 0.112 | 0.075 |
| 52 | 0.023 | 0.370 | 0.029 | -1.102 | 0.273 | 0.027 | -0.282 | 0.011 | 0.002 | 1.042 | 0.086 | 0.024 |
| 53 | 0.023 | 0.582 | 0.013 | 0.239 | 0.030 | 0.001 | -0.906 | 0.263 | 0.019 | 1.255 | 0.289 | 0.036 |
| 54 | 0.023 | 0.715 | 0.069 | 2.665 | 0.701 | 0.165 | -0.304 | 0.006 | 0.002 | -0.474 | 0.008 | 0.005 |

nim

| 61 | 0.023 | 0.265 | 0.014 | -0.352 | 0.060 | 0.003 | -0.481 | 0.069 | 0.005 | 0.896 | 0.137 | 0.018 |
| 62 | 0.024 | 0.835 | 0.003 | -0.399 | 0.398 | 0.004 | -0.408 | 0.256 | 0.004 | 0.433 | 0.181 | 0.005 |
| 63 | 0.024 | 0.013 | 0.002 | -0.034 | 0.005 | 0.000 | -0.043 | 0.004 | 0.000 | -0.050 | 0.004 | 0.000 |
| 64 | 0.020 | 0.679 | 0.020 | 0.914 | 0.242 | 0.017 | 1.081 | 0.209 | 0.024 | -1.495 | 0.228 | 0.045 |

eff

| 71 | 0.022 | 0.435 | 0.028 | -1.214 | 0.343 | 0.033 | 0.482 | 0.033 | 0.005 | 0.847 | 0.059 | 0.016 |
| 72 | 0.024 | 0.643 | 0.010 | -0.818 | 0.457 | 0.016 | -0.378 | 0.060 | 0.003 | 0.723 | 0.126 | 0.013 |
| 73 | 0.024 | 0.075 | 0.010 | 0.052 | 0.002 | 0.000 | -0.403 | 0.073 | 0.004 | -0.378 | 0.000 | 0.000 |
| 74 | 0.020 | 0.700 | 0.053 | 2.257 | 0.567 | 0.103 | 0.417 | 0.012 | 0.004 | -1.758 | 0.121 | 0.062 |

past50

<p>| 81 | 0.016 | 0.743 | 0.015 | -0.781 | 0.204 | 0.010 | 0.734 | 0.110 | 0.009 | -1.910 | 0.429 | 0.060 |
| 82 | 0.024 | 0.618 | 0.004 | -0.554 | 0.555 | 0.007 | -0.088 | 0.009 | 0.000 | -0.293 | 0.055 | 0.002 |</p>
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**Table 5.12.** Correspondence analysis coordinates and quality metrics for a 2 year horizon
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Table 5.14.

Correspondence analysis coordinates and quality metrics for 1 year horizon
coord

dimension 1
sqcorr

0.054
0.012
0.009
0.036

0.187
0.036
-0.081
-0.077

0.005
0.001
0.007
0.002

0.775
0.858
0.852
0.880

0.020
0.011
0.004
0.038

1.146
0.975
0.245
-1.717

0.421
0.827
0.175
0.870

0.018
0.024
0.025
0.024

0.575
0.238
0.533
0.502

0.035
0.008
0.008
0.020

0.141
0.047
0.014
-0.165

0.004
0.003
0.000
0.014

41
42
43
44

0.021
0.023
0.023
0.023

0.476
0.717
0.036
0.776

0.039
0.017
0.011
0.077

1.433
1.030
0.033
-2.456

0.470
0.623
0.001
0.733

51
52
53
54

0.022
0.022
0.023
0.024

0.646
0.550
0.591
0.789

0.067
0.029
0.013
0.085

1.727
1.127
-0.089
-2.600

0.412
0.414
0.006
0.779

61
62
63
64

0.022
0.024
0.024
0.021

0.288
0.855
0.045
0.698

0.013
0.003
0.002
0.020

0.540
0.404
-0.029
-1.014

0.211
0.540
0.005
0.445

71
72
73
74

0.022
0.024
0.024
0.021

0.456
0.722
0.093
0.777

0.027
0.011
0.010
0.055

1.153
0.811
0.011
-2.130

0.442
0.596
0.000
0.716

81
82

0.017
0.024

0.721
0.707

0.011
0.004

0.617
0.500

0.232
0.680

Categories

mass

overall
quality

%inert

11
12
13
14

0.017
0.023
0.025
0.026

0.586
0.257
0.401
0.530

21
22
23
24

0.016
0.023
0.025
0.027

31
32
33
34

contrib
rbc
0.001
0.000
0.000
0.000
ncl
0.020
0.022
0.002
0.080
equ
0.000
0.000
0.000
0.001
roa
0.044
0.025
0.000
0.137
cov
0.066
0.028
0.000
0.160
nim
0.007
0.004
0.000
0.021
eff
0.029
0.016
0.000
0.095
past30
0.006
0.006

coord

dimension 2
sqcorr

contrib

coord

dimension 3
sqcorr

contrib

3.467
0.692
-1.013
-1.918

0.550
0.128
0.394
0.382

0.206
0.011
0.026
0.096

1.081
0.898
0.044
-1.542

0.032
0.128
0.000
0.147

0.020
0.018
0.000
0.062

0.992
-0.320
-0.472
0.140

0.109
0.031
0.223
0.002

0.015
0.002
0.006
0.001

-1.934
-0.078
0.875
0.369

0.245
0.001
0.454
0.008

0.058
0.000
0.019
0.004

2.710
0.363
-1.057
-1.245

0.532
0.054
0.531
0.269

0.130
0.003
0.028
0.037

0.940
0.862
-0.075
-1.461

0.038
0.182
0.002
0.220

0.016
0.018
0.000
0.052

0.255
-0.401
-0.210
0.389

0.005
0.033
0.013
0.006

0.001
0.004
0.001
0.003

0.110
0.714
0.354
-1.208

0.001
0.061
0.022
0.036

0.000
0.012
0.003
0.033

1.341
-0.499
-0.894
0.079

0.086
0.028
0.198
0.000

0.040
0.006
0.018
0.000

-2.293
1.273
1.623
-0.624

0.149
0.108
0.388
0.009

0.116
0.036
0.060
0.009

-0.500
-0.432
-0.020
1.062

0.063
0.213
0.001
0.168

0.006
0.004
0.000
0.023

0.306
0.387
0.175
-0.980

0.014
0.101
0.039
0.085

0.002
0.004
0.001
0.020

0.352
-0.438
-0.370
0.558

0.014
0.060
0.050
0.017

0.003
0.005
0.003
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-0.013
0.595
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-1.178

0.000
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0.000
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0.005
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0.704
-0.116

0.104
0.013

0.008
0.000

-1.759
-0.157

0.385
0.014

0.052
0.001

305


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### Table 5.32. Eigenvalues and percentages of variance explained by PCA at Stage I

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### Table 5.34. Scoring coefficients for PCA at Stage I

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<td>0.7071</td>
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### Table 5.37. Eigenvalues and percentages of variance explained by PCA at Stage II

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<td>0.2090</td>
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<td>.0148967</td>
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<td>.269582</td>
<td>0.1642</td>
<td>0.7857</td>
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<td>.145647</td>
<td>0.1193</td>
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</table>
### Table 5.38: Scoring coefficients for PCA at Stage II

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<th>Comp3</th>
<th>Comp4</th>
<th>Comp5</th>
<th>Comp6</th>
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<td>-0.0217</td>
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<td>-0.0149</td>
<td>0.2700</td>
<td>0.6591</td>
<td>-0.2151</td>
</tr>
<tr>
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<td>0.4204</td>
<td>0.5569</td>
<td>-0.0008</td>
<td>-0.0697</td>
<td>0.6199</td>
<td>-0.3522</td>
</tr>
<tr>
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<td>0.1858</td>
<td>-0.0684</td>
<td>0.9575</td>
<td>-0.1359</td>
<td>0.0471</td>
</tr>
<tr>
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<td>0.4759</td>
<td>0.0001</td>
<td>0.0044</td>
<td>0.2956</td>
<td>0.6295</td>
</tr>
<tr>
<td>lgr</td>
<td>0.0047</td>
<td>0.0052</td>
<td>0.9975</td>
<td>0.0697</td>
<td>0.0007</td>
<td>-0.0009</td>
</tr>
</tbody>
</table>

### Table 5.40: Eigenvalues and percentages of variance explained by PCA at Stage I

Principal components/correlation

<table>
<thead>
<tr>
<th>Component</th>
<th>Eigenvalue</th>
<th>Difference</th>
<th>Proportion</th>
<th>Cumulative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comp1</td>
<td>2.04119</td>
<td>0.350258</td>
<td>0.2041</td>
<td>0.2041</td>
</tr>
<tr>
<td>Comp2</td>
<td>1.69093</td>
<td>0.227441</td>
<td>0.1691</td>
<td>0.3732</td>
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<tr>
<td>Comp3</td>
<td>1.46349</td>
<td>0.460595</td>
<td>0.1463</td>
<td>0.5196</td>
</tr>
<tr>
<td>Comp4</td>
<td>1.00289</td>
<td>0.0465847</td>
<td>0.1003</td>
<td>0.6198</td>
</tr>
<tr>
<td>Comp5</td>
<td>0.956307</td>
<td>0.0621954</td>
<td>0.0956</td>
<td>0.7155</td>
</tr>
<tr>
<td>Comp6</td>
<td>0.894112</td>
<td>0.189345</td>
<td>0.0894</td>
<td>0.8049</td>
</tr>
<tr>
<td>Comp7</td>
<td>0.704766</td>
<td>0.0718513</td>
<td>0.0705</td>
<td>0.8754</td>
</tr>
<tr>
<td>Comp8</td>
<td>0.632915</td>
<td>0.134184</td>
<td>0.0633</td>
<td>0.9387</td>
</tr>
<tr>
<td>Comp9</td>
<td>0.498731</td>
<td>0.384055</td>
<td>0.0499</td>
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<tr>
<td>Comp10</td>
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<td></td>
<td>0.0115</td>
<td>1.0000</td>
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</table>

Number of obs    = 273101
Number of comp.  = 10
Trace            = 10
Rotation: (unrotated = principal)  Rho              = 1.0000

### Table 5.41: Scoring coefficients for PCA at Stage III

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<th>Variable</th>
<th>Comp1</th>
<th>Comp2</th>
<th>Comp3</th>
<th>Comp4</th>
<th>Comp5</th>
<th>Comp6</th>
<th>Comp7</th>
<th>Comp8</th>
<th>Comp9</th>
<th>Comp10</th>
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<tbody>
<tr>
<td>rbc</td>
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<td>0.3369</td>
<td>-0.0051</td>
<td>-0.0293</td>
<td>-0.0016</td>
<td>-0.0137</td>
<td>0.0414</td>
<td>-0.0306</td>
<td>-0.0406</td>
<td>0.7184</td>
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<tr>
<td>ncl</td>
<td>-0.2710</td>
<td>0.5123</td>
<td>0.0762</td>
<td>0.1293</td>
<td>-0.1307</td>
<td>-0.0276</td>
<td>0.0386</td>
<td>0.6266</td>
<td>-0.4791</td>
<td>-0.0099</td>
</tr>
<tr>
<td>roa</td>
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<td>0.2465</td>
<td>0.3069</td>
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<td>0.1454</td>
<td>-0.1121</td>
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<td>0.0538</td>
<td>0.0891</td>
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</tr>
<tr>
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<td>-0.0702</td>
<td>-0.0083</td>
<td>0.9655</td>
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<tr>
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<td>-0.3367</td>
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<td>-0.1271</td>
<td>-0.0516</td>
<td>0.8850</td>
<td>0.3351</td>
<td>-0.1374</td>
<td>0.0468</td>
<td>0.0552</td>
<td>0.0024</td>
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<tr>
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<td>-0.3198</td>
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<td>0.0004</td>
<td>-0.4683</td>
<td>-0.4119</td>
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<td>-0.0567</td>
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<td>0.5877</td>
<td>0.1352</td>
<td>0.1749</td>
<td>0.4054</td>
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